GRAIN FLOW THEORY AND SNOW AVALANCHE RHEOLOGY

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ABSTRACT: This paper explores the application of grain flow theory to snow avalanche dynamics, focusing on the interrelation between shear stress and fluctuation energy. The study employs a system of differential equations to model the mean velocity (U) and fluctuation energy (R) within snow avalanches. The key physical constraint highlighted is the relationship $dR \cdot S = -R_0 \cdot dS$, which provides a physical constraint on the complex relationship between avalanche shear stress and the production of fluctuation energy. When fluctuation energy is produced by shearing, it generates not only directionless random kinetic energy but also directional energy fluxes that alter the configuration of the granular ensemble, leading to changes in shear stress. The analysis of the R-U phase plane reveals that avalanche behavior can be characterized by equilibrium points that shift with slope angle, highlighting the role of fluctuation energy in determining the flow regime. The R-U phase plane not only advances our understanding of avalanche dynamics but also facilitates the examination of flow regime transitions, driven by changes in the potential energy of the granular ensemble, and the emergence of different avalanche types.

Keywords: Avalanche dynamics, shear stress, grain flow theory, flow regime, flow regime transitions.

1. INTRODUCTION

When grain flow theory (Haff, 1983) is applied to model snow avalanche flow (Fig. 1), an additional differential equation is added to the momentum balance, resulting in a system of differential equations for the mean velocity U and particle fluctuation energy R (granular temperature)

$$\begin{aligned} \frac{dU(t)}{dt} &= \dot{U}(t) = G - S(R) \\ \frac{dR(t)}{dt} &= \dot{R}(t) = \alpha \dot{W}_f(S,U) - \beta R \end{aligned} \qquad (1)$$

In these equations G is the gravitational acceleration in the slope parallel direction; $S(R)$ the frictional resistance which is some function of R and $\dot{W}_f(S,U)$ is the frictional work rate. The second equation can be considered a logistic-type equation as it statistically governs the production (parameter α) and decay (parameter β) of the granular fluctuation energy from the shear work rate (Haff, 1983; Jenkins and Savage, 1983; Gubler 1986; Buser and Bartelt, 2009). The production can also be linked to the vertical dispersion (dilitancy) of the avalanche (Reynolds, 1885). Part the produced fluctuation energy is reversed at the basal boundary, leading to rapid changes in flow height and particle dispersion (Fig. 2). Moreover, changes in granular configuration and the avalanche flow density are directly related to the production of R from the shear rate. Different formulations for the decay of the fluctuation energy exist, depending on the size and elasticity of the particles involved in the flow.

The fluctuation energy R arises because the i-th particle in a volume with N particles each with mass m_i is moving at a speed u_i different than the mean velocity of the avalanche

$$\sum_i^N \frac{1}{2} m_i \left(u_i'\right)^2 \quad \text{with} \quad u_i' = u_i - U \qquad (2)$$

The fluctuation energy has the property

$$\sum_i^N u_i' = 0 \quad \text{and} \quad \sum_i^N \left(u_i'\right)\left(u_i'\right) \neq 0 \qquad (3)$$

Thus, the first moment vanishes while the second moment of the granular fluctuations remains non-zero. It represents a granular state variable that can be exploited to describe the flow state of the avalanche. For example, the system of differential equations (Eq. 1) is *autonomous*, as both equations do not directly explicitly on the time t and are coupled by the avalanche friction $S(R)$ and friction work rate $\dot{W}_f(S,U)$. With grain flow theory it is possible to examine avalanche behaviour on the R-U phase plane (Bartelt, 2011) to define avalanche flow regimes, which we define as a specific equilibrium point (R, U) in the R-U phase plane. That is, the introduction of grain flow theory allows us to model both fluidized avalanche flow states (typified by high R and high U) and dense regimes (marked by low R and low U). These flow states can even exist simultaneously in the same avalanche, for example a fluidized avalanche regime at the avalanche front,

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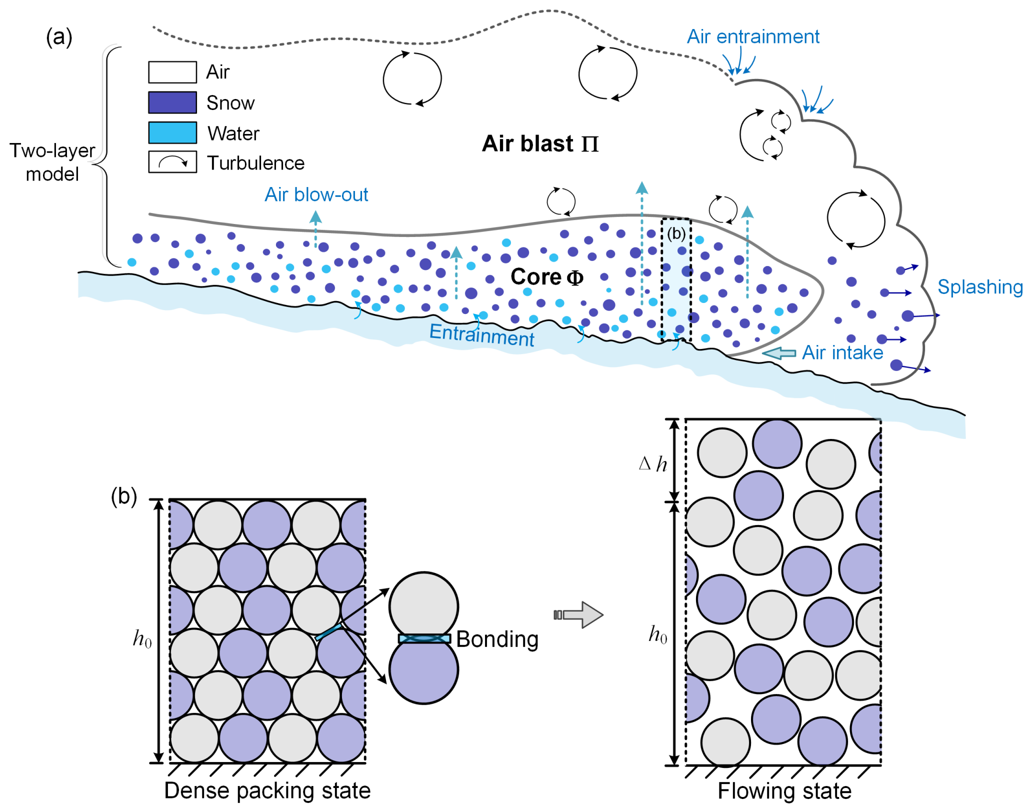

Figure 1: The avalanche core consists of a flowing granular ensemble of snow particles. The particles disperse creating distinct configurations and flowing regimes, such as dense (frictional) or disperse (collisional). Energy fluxes associated with changes of particle configurations are linked to changes in avalanche shear strees.

and a dense avalanche regime at the avalanche tail. More significantly, we can define the physical conditions for flow regime transitions and the emergence of different avalanche types, such as the formation of a powder avalanche.

The mechanical energy flux $\dot{R}$ is generated by shearing, but not dissipated immediately to heat. It represents an intermediate energy form, that arises from kinetic movement and shearing, but not yet dissipated to heat. It is of interest that the competition between the intermediate energy form (R) and final energy form heat (temperature T) is a long-standing idea in avalanche dynamics. Voellmy (1955) recognized early that two highly mobile avalanche types existed, dry fluidized avalanche (R-avalanches) and wet snow avalanches (T-avalanches).

2. GRAIN FLOW FRICTION S(R)

Avalanche flow regimes are characterized by the configuration of the N snow particles within a given volume V. We define the co-volume V_0 (see Figure 1) as the densest possible packing of the particles, which corresponds to a dense flow regime (Fig. 1). Fluidized flow regimes, on the other hand, are associated with dispersed granular configurations. The primary distinction between these two states lies in their respective potential energies. Denoting the position of the i-th particle as z_i, the potential energy P of the granular ensemble is given by:

$$P = \sum_{i=1}^{N} m_i g z_i \tag{4}$$

where the reference configuration is taken from the basal running surface of the avalanche. We term P the potential energy virial, as it helps to describe the equilibrium conditions and the energy fluxes in an ensemble of particles. Any change in the z-positions of the particles over time corresponds to a change in potential energy and therefore a change in the particle configurations:

$$\frac{dP}{dt} = \sum_{i=1}^{N} m_i g \frac{dz_i}{dt} \tag{5}$$

It is natural to associate changes in particle configurations with avalanche flow regimes. The change in potential energy is driven by the input of fluctuation energy at the basal boundary of the avalanche, a flux of a random kinetic energy. Moreover, when fluctuation energy dR is introduced in the flowing granular ensemble of snow particles, there is a change in shear stress. This change is such that as the potential energy increases the shearing resistance decreases. To mathematically account for this process we place the following condition on the

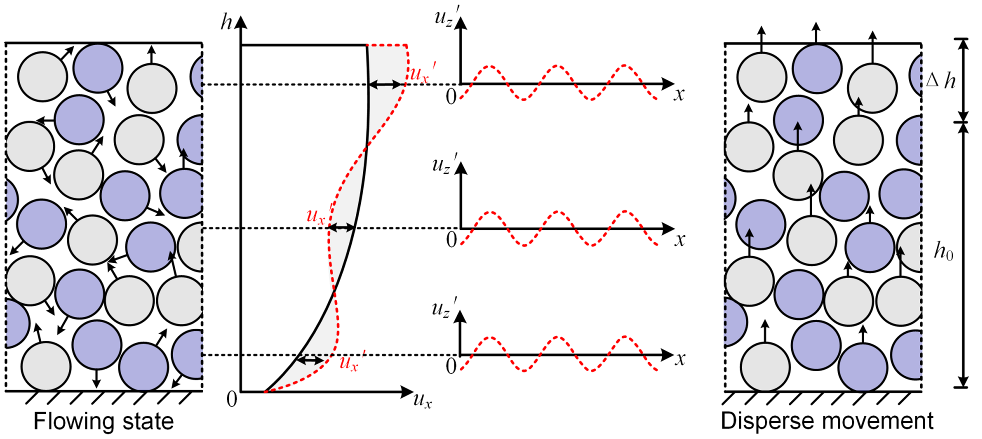

Figure 2: Random velocity fluctuations exist in the slope parallel x and slope perpendicular z directions. Because of the basal boundary conditions and the upper free surface the random fluctuations become directional, re-organizing the granular configuration and changing the avalanche shear stress.

relationship between fluctuation energy and shearing: *For a given perturbation of macroscopic fluctuation energy dR imposed at a stress level S, the change in shearing resistance dS is proportional to the product $dR \cdot S$ by the constant R_0* ,

$$dR \cdot S = -R_0 \cdot dS. \tag{6}$$

This equation describes the fluctuation energy required to fluidize a granular ensemble and change its resistance to shearing. The condition necessarily leads to stable flow regimes that can only change under external perturbations, such as changes in avalanche boundary conditions such as slope, snow temperature or entrainment conditions. The relationship implies that the fluctuation energy needed to fluidize the granular system decreases with increasing macroscopic fluctuations R, leading to a shear stress S that increases as the macroscopic fluctuations decrease. In more simple terms, the shear stress increases as the flow regime becomes more dense (frictional), and decreases as the flow regime becomes more disperse (collisional). The minus sign in Eq. 9 ensures that when the fluctuation energy increases (positive dR), the shear stress decreases, whereas if the fluctuation energy decreases (negative dR), the shear stress increases. This indicates that the shearing resistance decreases with increasing fluctuations, making the snow granule ensemble more susceptible to fluidization and changes in shear stress.

The idea of Eq 9 is grounded in the idea that there is only one source of energy for an avalanche: the gravitational work rate, $\dot{W}_g$. The three primary energy fluxes (the change in translational kinetic energy $\dot{K}$, the frictional work rate $\dot{W}_f$ and production ad decay of fluctuation energy $\dot{R}$ must change at the expense of the other,

$$\dot{W}_g \rightarrow \dot{K} \overset{S}{\rightleftharpoons} \dot{W}_f \overset{S}{\rightleftharpoons} \dot{R} \tag{7}$$

This equation implies that the energy allocation among these fluxes is competitive in nature; an increase in one must be balanced by a decrease in the others. Since both shearing and fluctuations draw from the same energy reservoir, their competition is inherently constrained. The system, through this interaction, evolves towards a configurational equilibrium—a dynamic flow regime characterized by the balance of these competing processes. This equilibrium represents a state in which the energy fluxes achieve a stable configuration: a flow regime. This relationship 9 is written in incremental form because it is valid for all changes in dR for any state variable used to describe the system, for example in time t,

$$\frac{dR}{dt} S = -R_0 \frac{dS}{dt} \tag{8}$$

or, for example, in changes of the potenial virial P,

$$\frac{dR}{dP} S = -R_0 \frac{dS}{dP}. \tag{9}$$

The relationship Eq. 9 can be re-written,

$$\frac{dS(R)}{dR} = -\frac{S(R)}{R_0} \tag{10}$$

which implies

$$S(R) = S_0 \exp^{-\frac{R}{R_0}} \tag{11}$$

where S_0 is the frictional resistance of the co-volume. We take a Voellmy relationship

$$S_0 = \mu_0 N + \frac{\rho g U^2}{\xi_0} \quad S(R) = \left[\mu_0 N + \frac{\rho g U^2}{\xi_0}\right] \exp^{-\frac{R}{R_0}} \tag{12}$$

The activation energy R_0 has several physical meanings

1. *Characteristic Scaling*. The parameter R_0 can be interpreted as a characteristic energy

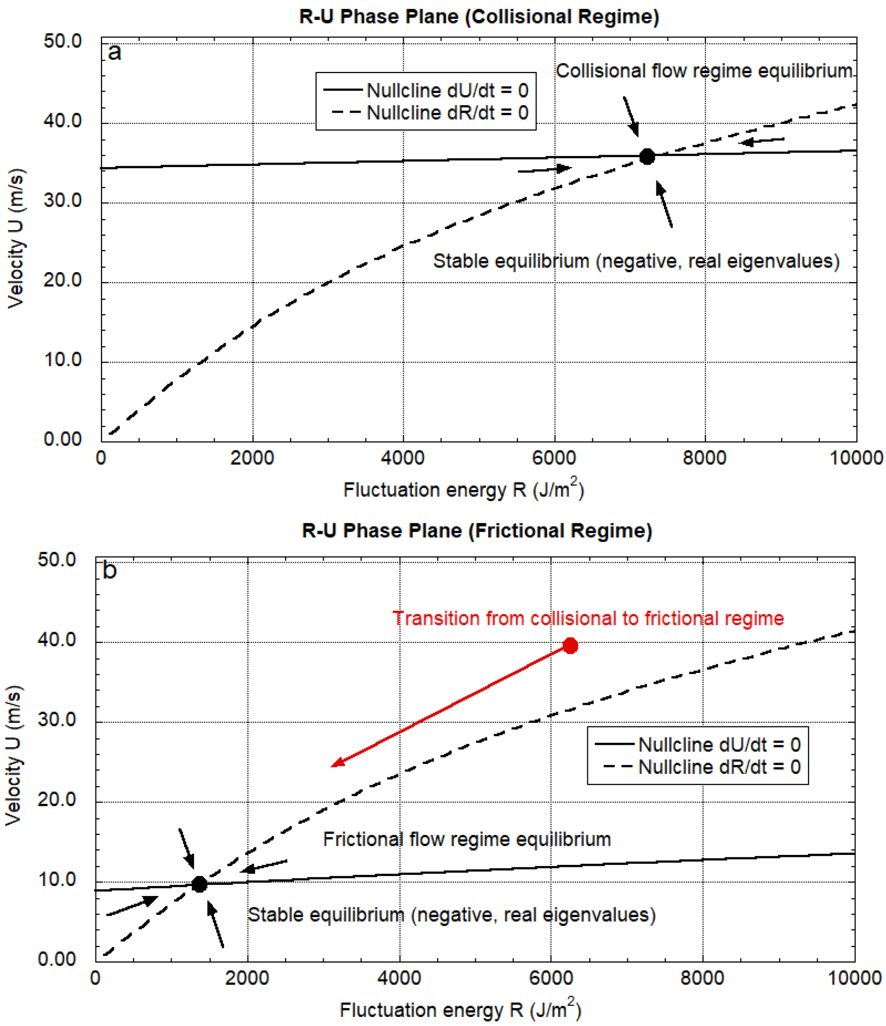

Figure 3: The R-U phase plane of a snow avalanche reveals the intricate dynamics of flow regimes under different slope angles. a) For a steep slope angle, the R-U phase plane displays a state of flow equilibrium characterized by high velocity and substantial fluctuation energy. This elevated fluctuation energy is associated with a disperse flow regime, leading to the formation of powder avalanches. The system's trajectory reflects a dynamic equilibrium where avalanche speed (U) and fluctuation energy (R) interact under high-energy conditions. b) At a lower slope angle, the R-U phase plane shows a transition to a different equilibrium state with lower fluctuation energies and velocities. This results in the emergence of a dense flow regime. As the avalanche descends, the system transitions from the high-energy state depicted in figure a to the more stable, lower-energy state observed in figure b. This transition is characterized by changes in the nullclines that define the flow trajectories, indicating how the relationship between avalanche speed (U) and fluctuation energy (R) evolves as the avalanche moves downhill.

scale associated with the system's response to macroscopic fluctuations. It represents the scale at which the shear stress decays significantly in response to changes in the macroscopic structure or fluctuations.

2. *Sensitivity Factor*. R_0 also serves as a sensitivity factor that determines how quickly the shear stress diminishes with increasing macroscopic fluctuations. A smaller R_0 indicates a faster decay in shear stress with fluctuations, while a larger R_0 implies a more gradual decrease. S

3. *Stability Indicator*. The value of R_0 influences the stability of the system. A smaller R_0 may lead to rapid changes in shear stress with fluctuations, potentially indicating instability, while a larger R_0 suggests more robust stability as the system responds more gradually to fluctuations.

The stability of the avalanche flow regime (frictional, collisional) can be found by investigating the eigenvalues of the Jacobian matrix of Eq. 1,

$$\mathbf{J} = \begin{pmatrix} \frac{\partial \dot{U}}{\partial U} & \frac{\partial \dot{U}}{\partial R} \\ \frac{\partial \dot{R}}{\partial U} & \frac{\partial \dot{R}}{\partial R} \end{pmatrix} \tag{13}$$

The elements of the Jacobian matrix physically represent gradients of the dissipative energy fluxes (shearing, decay of R) in the R-U plane. It can be easily shown that both eigenvalues of $\mathbf{J}$ using the Voellmy shear stress (Eq. 12) are real and negative, indicating stability for all U and R,

$$\lambda_1, \lambda_2 = \frac{A \exp^{-\frac{R}{R_0}} - \beta R_0 \pm \sqrt{B}}{2R_0} \tag{14}$$

with the constants A, B provided in the appendix.
The stability depends on the magnitude of R_0. For every slope angle, a different equilibrium is found indicating that the avalanche will transit through different flow regime varying with disperse flow regimes on steep slopes to frictional flow regimes on flat slopes (Figure 2). In the R-U phase plane, the equilibrium points are attractors suggesting that as the avalanche transits from one slope segment to the next, the avalanche will undergo flow regime transitions, before reaching a new stable flow regime. This flow regime is found at the intersection of the nullclines of Eq. 1.
The velocity fluctuations in the slope-parallel x-direction deviate from the mean velocity U (Fig. 3a). However, the fluctuations in the slope-perpendicular z-direction interact with the hard basal boundary (Fig. 3b), triggering an energy exchange between the random fluctuation energy R and the potential energy P of the granular ensemble. This interaction at the basal boundary directs the random energy flux upward, expanding the avalanche flow surface and forming powder avalanches. The basal boundary organizes the random energy into a directional flux, leading to the emergence of distinct granular configurations with varying spacing and structures. These self-organizing processes define specific avalanche flow regimes, which can be represented in $R - U$ space. The relationship $dR \cdot S = -R_0 \cdot dS$ mathematically captures the self-organization process, illustrating how velocity fluctuations lead to new flow equilibria and the associated shearing stress within the granular system.

3. ACTIVATION ENERGY R_0 = 2 kJ/m^3

To demonstrate how grain flow theory is applied in practice we consider a small (V_0 = 7'000m^3) avalanche that occurred on January 15, 2019 on the Masura avalanche track near Klosters, Switzerland (Fig. 3). The avalanche destroyed trees in a 50 year old forest, leaving avalanche deposits on the winter hiking path. Because of the track's proximity to Davos it was possible to document the release zone, snow conditions and forest damage.
We simulate this avalanche with the RAMMS::EXTENDED avalanche model, specifying an activation energy of R_0=2kJ/m^3. The calculated avalanche runout with powder cloud (Fig 5a), fluctuation energy (Fig. 5b), Coulomb friction (Fig. 5c) and turbulent friction (Fig 5d) are shown in Fig. 5. Because of the cold conditions the avalanche entered a mixed flowing avalanche regime with a strong decrease in Coulomb friction

Figure 4: Case study Masura, 15.01.2019 Klosters. An avalanche with an estimated return period of 30 years formed on the steep, cold (northern exposure) avalanche track. The picture was taken one-day after the avalanche event.

(from μ_0 = 0.55 at R = 0 to $\mu(R) \approx 0.15$ at R=5 kJ/m^3 when the avalanche reached the maximum flow velocity. When the avalanche entered the deposition zone, with much lower slope angle, the Coulomb friction decreased to $\mu(R) \approx 0.30$ at R=2 kJ/m^3 (Fig. 5c) A similar change in ξ is also produced. The grain flow model reproduces the both the flow width of the core as well as the destructive force of the powder cloud.

4. CONCLUSION

In examining snow avalanche dynamics through the lens of grain flow theory, this study reveals a connection between fluctuation energy and shear stress, encapsulated by the constraint $dR \cdot S = -R_0 \cdot dS$. This relationship underscores how changes in fluctuation energy influence the shear resistance of the avalanche, providing a mathematical basis for understanding the transitions between different avalanche flow regimes. The analysis of the R-U phase plane further elucidates the impact of slope

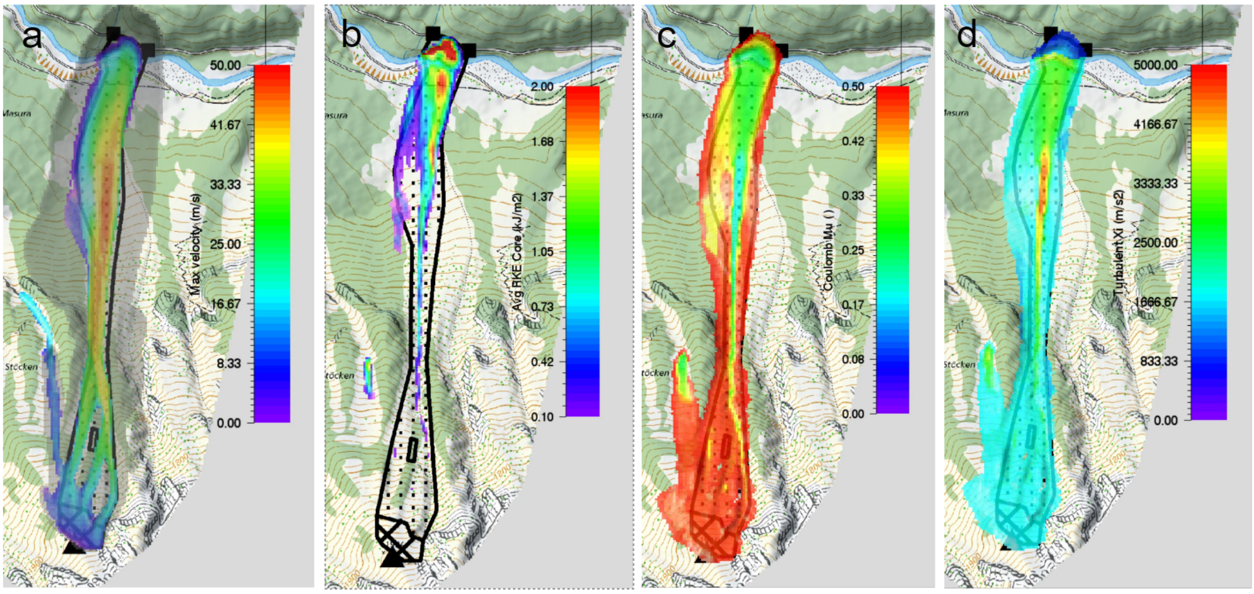

Figure 5: Case study Masura, 15.01.2019 Klosters. Avalanche calculated with RAMMMS::Extended. a) Max avalanche flow velocity and powder cloud (gray). b) Depth-averaged fluctuation energy R at t =45s. Max R is around 2kJ/m^2 in the runout zone. c) Calculated $\mu(R)$. Coulomb friction is the deposition zone is approximately=0.30 $\mu(R)$=0.30 with μ_0 = 0.55. d) Voellmy friction $\xi(R)$.

angle on avalanche behavior, illustrating how variations in energy and velocity dictate the formation of fluidized powder avalanches and dense flowing avalanche flow configurations.

Importantly, these macroscopic flow states have significant implications for various avalanche characteristics, including avalanche deposits, runout distances, the formation of powder avalanches, and even the magnitude of avalanche impact pressures. By characterizing these flow regimes through the constraint $dR \cdot S = -R_0 \cdot dS$, this framework highlights the interplay between granular fluctuations and macroscopic flow characteristics. The relationship governing the positive perturbation of fluctuation energy is countered by a fall in the shear stress that ensures a converging feedback, where the avalanche always reaches a stable equilibrium without unlimited growth in fluctuation energy.

Thus, the interplay between velocity fluctuations and changes in shear stress is central to the self-organization of the avalanche, leading to the emergence of distinct flow regimes and ensuring that the system evolves toward a stable equilibrium state.

5. APPENDIX

The eigenvalues of the Jacobian matrix for a grain flow Voellmy relationship are found using the symbolic calculation tool Maple (6). The values for A and B are given below:

$$A = -\left[\alpha\xi'\rho h\, U^2 + \alpha c_1 h\rho + 2R_0\xi'\right] U \tag{15}$$

$$B = (B_1 + B_2 + B_3)\exp^{-\frac{2R}{R0}} + 2\beta U\,(C_1 + C_2)\,R_0 \tag{16}$$

$$B_1 = [\alpha\rho h U]^2 \left(\xi'\, U^2 + \mu'\right)^2 \tag{17}$$

$$B_2 = 8\alpha\rho h\left(\xi'\, U^2 + \frac{\mu'}{2}\right)\left(\xi'\, U^2 + \mu'\right) R_0 \tag{18}$$

$$B_3 = 4\,[\xi' U R_0]^2 \tag{19}$$

$$C_1 = \left[\alpha\rho h\left(\xi'\, U^2 + \mu'\right) - 2R_0\xi'\right]\exp^{-\frac{R}{R_0}} \tag{20}$$

$$C_2 = \frac{\beta R_0}{2} \tag{21}$$

$$\mu' = \mu_0 g\cos(\theta) \qquad \xi' = \frac{g}{\xi_0 h} \tag{22}$$

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