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# Gravity Override of Immiscible Fluids in Porous Media


## Abstract

*When one fluid is displaced by another in a permeable medium, the front between the two fluids changes its shape. The change in shape of the front depends on miscibility of the two fluids, the density difference, and the time of contact. Inside a petroleum reservoir, when oil is pushed with a fluid, the front's changed shape is one of the important factors in determining the efficiency of the production system. Therefore, it is important to study the changing shape of the front, in order to extract oil in an efficient manner. Using Darcy's law and the convection-diffusion equation, a detailed mathematical study of the motion of the front and the fluid particles is presented for miscible fluids. The model is then restricted to immiscible fluids and numerical results are presented. This paper describes how the front between two fluids changes its shape because of a difference in density.*


**Key Words**: fluid flow, porous media, immiscible fluid, miscible fluid.

## Introduction

The phenomena of the simultaneous flow of two fluids through a porous medium occurs in many important problems of petroleum recovery, reservoir engineering and groundwater hydrology. Consequently, many problems have arisen and been solved in past years. Problems in hydrology and geology related to the flow of two fluids concern the more or less natural contact of two fluids inside the earth. However, in petroleum reservoir engineering the contact of the fluids is an artificial situation created in the oil field inside the reservoir. Examples are pushing the oil (with miscible or immiscible fluids) toward the production well to get more output, the under ground motion of a spilled contaminant that is moving towards a river or a lake, and seawater intrusion into a coastal freshwater aquifer.

The problem discussed here comes from an oil field where a fluid is injected through an injection well to increase the pressure inside the reservoir and to push the oil toward the production wells. When the injection fluid is injected at a very high pressure the pressure is distributed in the flow domain and the front between the two fluids (oil and injection fluid) moves toward the production wells. The motion and shape of the front between two fluids depends on the contrast of the fluids such as high or low density difference, high or low viscosity difference, and high or low miscibility of the fluids. Initially the front between the two fluids can be assumed to be sharp and vertical. After awhile, because of miscibility, the front is not sharp anymore as shown in Figure 1. Using Darcy's law and the convection- diffusion equation, an extensive mathematical study of the motion of the front between two fluids is possible. A mathematical model has been developed that studies the change in the shape of the front, under gravitational force only, inside a porous medium. It has been assumed that the fluid particles near the front are in motion only due to density differences of the two

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fluids. The density difference between the two fluids causes a pressure gradient which makes the front move.

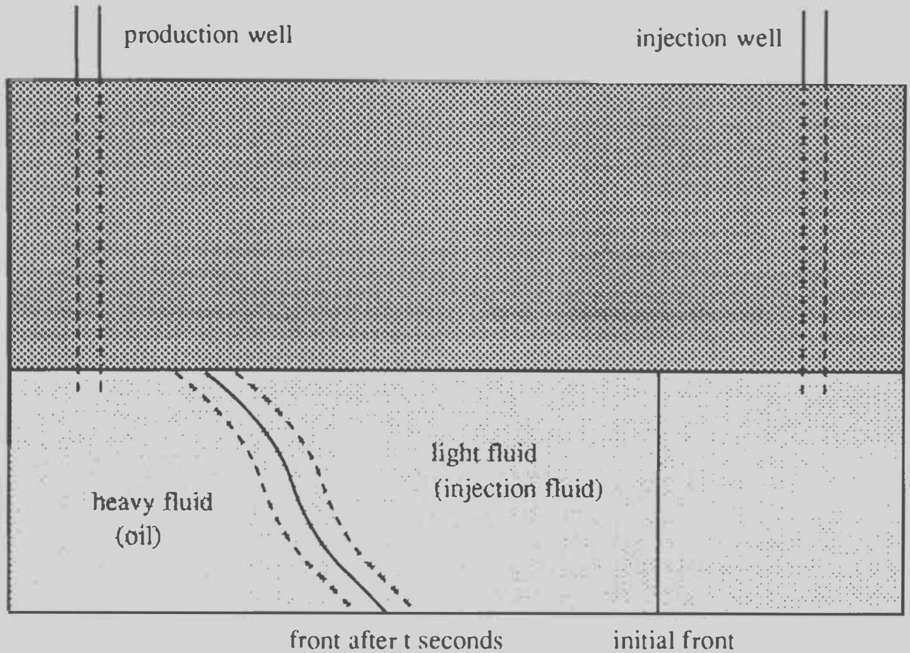



**Figure 1.** *Motion of the front inside the reservoir*

## Diffusion and Flow

If a liquid is in contact with another substance (solid or gas) there is free interfacial energy present between the two. This means that a certain amount of work has to be performed in order to separate a liquid from, say, a solid. The interfacial energy comes from the inward attraction of the molecules in the interior of a substance toward those at the surface. Since a surface possessing free energy contracts (if it can do so), the free interfacial energy manifests itself as *interfacial tension*, (Collins 1976). Thus the interfacial tension $\sigma_{ik}$ for a pair of substances $i$ and $k$ is defined as the amount of work that must be performed to separate a unit area of substance $i$ and substance $k$. For air and water, at 20°C, $\sigma_{ik} = 72.5 erg/cm^2$. The interfacial tension $\sigma_i$ between a substance $i$ and vacuum (or vapor of the same substance) is called *surface tension*.

If the interfacial tension between two fluids is nonzero, the fluids do not mix. This is referred to as immiscible. If the interfacial tension between two fluids is zero then a distinct fluid interface does not exit and the fluids are miscible. If two fluids are miscible then molecules of one fluid can diffuse into the other fluid. This is a spontaneous process. Consider two fluids brought into contact at a plane. Within either fluid the molecules have a random motion which is dependent upon the absolute temperature. This motion is isotropic, *i.e.* in any homogeneous region there are equal numbers of molecules moving in all directions with the same distribution of velocity. At the plane of separation there are molecules of *kind 1* on the left and molecules of *kind 2* on the right. Due to random motion some molecules of *kind 1* cross the plane to the right and some of *kind 2* cross to the left. This process expands in both directions until a homogeneous mixture of two kinds of molecules exists. This process is termed *molecular diffsion* (Rafai 1956).

## Convection-Diffusion Equation

Assume a linear fluid element between A and B. If C is the average concentration of the fluid at a certain instant, then

$$\text{Total mass} = \int_A^B C\,dx. \qquad (1)$$

Since mass is conserved, the rate of change of mass is equal to the change in mass per unit time due to flow plus the change in mass per unit time due to diffusion. That is

$$\frac{\partial}{\partial t}\int_A^B C dx = (vC(A) - vC(B) + (DC_x(B) - DC_x(A)$$

where D is the diffusion coefficient. But

$$(vC(A) - vC(B)) = -\int_A^B (vC)_x dx$$

and

$$(DC_x(B) - DC_x(A)) = \int_A^B (DC_x)_x dx.$$

This gives

$$\int_A^B C_t dx + \int_A^B (vC)_x dx - \int_A^B (DC_x)_x dx = 0$$

and so

$$\int_A^B (C_t + (vC)_x - (DC_x)_x) dx = 0.$$

If the material can be regarded as having continuous properties then

$$C_t + (vC)_x - (DC_x)_x = 0.$$

In two dimensions this becomes

$$C_t = -\nabla\cdot(\vec{v}C) + \nabla\cdot(D\nabla C). \qquad (2)$$

## Darcy's Equation

In 1856 Henry Darcy described in an appendix of his book *"La Fountaina Publiques de la Ville de Dijon"* a series of experiments on the downward flow of water through sand filters, whereby it was established that the rate of flow depends on the pressure difference between the two end points and the length and the cross-sectional area of the sand pack (Hubbert 1969). The liquid used in Darcy's original experiment was water. Later, the experiment was repeated using different liquids and a generalized relation between the flow rate, density and viscosity of the liquid, and porosity, permeability, length, and cross-sectional area of the sand pack was established.

**Definition 0.1 Porosity** *The porosity of a porous material is the fraction of the bulk volume of the material occupied by the voids. The symbol usually employed for this parameter is $\phi$. Thus*

$$\phi = \frac{V_p}{V_B} = \frac{\text{Volume of the pores}}{\text{Bulk volume}}.$$

*The porosity $\phi$ is a dimensionless quantity.*

Two kinds of porosity can be defined, namely absolute or total porosity, and effective porosity. Absolute porosity is the fractional void space with respect to bulk volume regardless of pore connections. Effective porosity is that fraction of the bulk volume constituted by interconnecting pores. Many naturally occurring rocks, such as lava and other igneous rocks, have a high total porosity but essentially no effective porosity. The porosity can range from zero to more than 0.5. In most sedimentary rocks, however, the porosity is predominantly interconnected and seldom exceeds 0.3.

**Definition 0.2 Permeability** *This is the property of the porous material which characterizes the ease with which a fluid may be made to flow through the material by an applied pressure gradient. Permeability is the fluid conductivity of the porous material.*

The permeability of the medium measures the facility with which fluids flow through the medium. For the present only a quantitative definition is necessary. Of two media through which the same fluid is made to flow under identical conditions, that the medium through which the flow is more rapid has the greater permeability. A medium may be said to be isotropic with respect to permeability if it is equally permeable in all directions.

Inside the porous medium, where $\vec{v}$ is the fluid velocity and $\phi$ is the porosity of the medium, Darcy's velocity $\vec{u}$ is defined by

$$\vec{v} = \frac{\vec{u}}{\phi}. \qquad (3)$$

If a liquid with density $\rho$ and viscosity $\mu$ is flowing through a porous medium which has permeability, $k$, then Darcy's velocity is given by (Muskat 1946 and Bear 1979)

$$\vec{u} = -\frac{k}{\mu}\left(\nabla P + \rho\vec{G}\right) \qquad (4)$$

and thus

$$\nabla P + \frac{\mu}{k}\vec{u} + \rho\vec{G} = 0 \qquad (5)$$

where $P$ is the pressure and $\vec{G} = (0,g)$ is the gravitational acceleration.

Darcy's experiment and the derivation of these equations is discussed in detail in Collins (1976).

## Model Development

### Another form of Darcy's Equation

From (5) one can take the divergence and, assuming that $k$ is constant, arrive at

$$\nabla^2 P + \frac{\nabla\mu\cdot\vec{u}}{k} + \frac{\mu}{k}\nabla\cdot\vec{u} + \nabla\rho\cdot\vec{G} + \rho\nabla\cdot\vec{G} = 0 \qquad (6)$$

Since the fluid is assumed to be incompressible, the divergence of its velocity is zero. This fact eliminates the third term of (6). Also since $\vec{G}$ is a constant vector, $\rho\nabla\cdot\vec{G} = 0$. This eliminates the fifth term of (6). Hence (6) reduces to

$$\nabla^2 P + \frac{1}{k}\nabla\mu\cdot\vec{u} + \nabla\rho\cdot\vec{G} = 0$$

But using (4) gives

$$\nabla^2 P - \frac{\nabla\mu}{\mu}\cdot(\nabla P + \rho\vec{G}) + (\nabla\rho\cdot\vec{G} = 0. \qquad (7)$$

Since $\frac{\nabla\mu}{\mu} = \nabla(\ln\mu)$, (7) becomes

$$\nabla^2 P - \nabla(\ln\mu)\cdot\nabla P - \rho\nabla(\ln\mu)\cdot\vec{G} + \nabla\rho\cdot\vec{G} = 0.$$

Since $\rho$ and $\mu$ are both functions of the concentration, $C$, this equation can be written as

$$\nabla^2 P - (\ln\mu)'\nabla C\cdot(\nabla P + \rho\vec{G}) + (\rho)'(\nabla C\cdot\vec{G}) = 0, \qquad (8)$$

where $\mu' = \frac{\partial\mu}{\partial C}$ and $\rho' = \frac{\partial\rho}{\partial C}$.

If two fluids with densities $\rho_1$ and $\rho_2$, viscosities $\mu_1$ and $\mu_2$, and concentrations $C_1$ and $C_2$ respectively, are diffusing into each other, then empirical results (Peaceman 1955) show that (*i*) the density $\rho$ of the mixture is a linear function of C and (*ii*) if $\mu$ is the viscosity of the mixture then $\ln(\mu)$ is a linear function of $C$, where $C$ is the concentration of fluid one. Thus

$$\rho = (\rho_1 - \rho_2)C + \rho_1, \qquad (9)$$

$$\ln(\mu) = (\ln(\mu_1) - \ln(\mu_2))C + \ln(\mu_1)$$

and hence

$$\ln(\mu) = \ln(M)C + \ln(\mu_1), \qquad (10)$$

where $M = \frac{\mu}{\mu_2}$. Hence (8) reduces to

$$\nabla^2 P - \ln M(\nabla C\cdot\nabla P) - (\rho\ln M - (\rho_1 - \rho_2))\left(\frac{\partial C}{\partial y}g\right) = 0. \qquad (11)$$

## Diffusion Equation and Velocity of the Sharp Front

From (2)

$$\frac{\partial C}{\partial t} = -(\nabla \cdot \vec{v})C - \vec{v} \cdot \nabla C + \nabla D \cdot \nabla C + D\nabla^2 C.$$

If we assume that the fluid is incompressible and that the diffusion coefficient is constant, then this equation reduces to

$$\frac{\partial C}{\partial t} = -\vec{v} \cdot \nabla C + D\nabla^2 C. \qquad (12)$$

If at a certain instant of time concentration and velocity are known then this equation can be solved numerically.

In the particular case where there is no mixing between the two fluids, then $D = 0$. The fluids are thus immiscible. Since $C$ is a function of position and time,

$$\frac{dC}{dt} = \frac{\partial C}{\partial t} + \nabla C \cdot \frac{d\vec{x}}{dt}, \qquad (13)$$

where $d\vec{x} = (dx, dy)$. Following a particle on a particular iso-concentration line, $\frac{dC}{dt} = 0$. Hence (13) gives

$$\frac{\partial C}{\partial t} = -\nabla C \cdot \frac{d\vec{x}}{dt} = -\nabla C \cdot \vec{V}, \qquad (14)$$

where $\vec{V}$ is the velocity of the front. From (12) and using the fact that $D = 0$,

$$\vec{V} \cdot \nabla C = v \cdot \nabla C$$

and so

$$\vec{V} \cdot \hat{n} = \vec{v} \cdot \hat{n}.$$

Here $n = \frac{\nabla C}{\|\nabla C\|}$ is the unit vector in the direction of $\nabla C$ which is also the direction of the velocity of the front. This implies that the speed, $V$, of the front is

$$V = \vec{v} \cdot \hat{n}. \qquad (15)$$

This is the same as the normal component of the velocity of the fluid particle on the front. Thus if the velocity of the fluid particles on the front is known then the speed of the front can be determined. By (3) and (4)

$$\vec{v} = -\frac{k}{\mu\phi}\left(\nabla P + \rho\vec{G}\right),$$

which gives

$$v_x = -\frac{k}{\mu\phi}\frac{\partial P}{\partial x} \qquad (16)$$

and

$$v_y = -\frac{k}{\mu\phi}\left(\frac{\partial P}{\partial y} + \rho g\right), \qquad (17)$$

where $v_x$ and $v_y$ are the horizontal and vertical components of $v$, respectively.

At a certain instant of time if the velocity components $v_x$ and $v_y$ are known and the inclination of the front from horizontal is $\alpha$ then the velocity of the front can be determined by the formula

$$V = v_x \sin\alpha + v_y \cos\alpha. \qquad (18)$$

## Initial Conditions

Consider a rectangular box of length $l$ and height $h$ (Fig. 2). Initially it is assumed that the front is vertical and in the middle of the box; the heavy fluid is in the left-hand side of the box and the light fluid is in the right-hand side. Initially we have

$$P(x,y) = \begin{cases} \rho_1 g(h-y) & x < l/2 \\ \rho_2 g(h-y) & x \geq l/2, \end{cases} \qquad (19)$$

$$C(x,y) = \begin{cases} 1 & x < l/2 \\ 0 & x \geq l/2, \end{cases} \qquad (20)$$

and

$$\rho(x,y) = \begin{cases} \rho_1 & x < l/2 \\ \rho_2 & x \geq l/2, \end{cases} \qquad (21)$$

$$\mu(x,y) = \begin{cases} \mu_1 & x < l/2 \\ \mu_2 & x \geq l/2. \end{cases} \qquad (22)$$

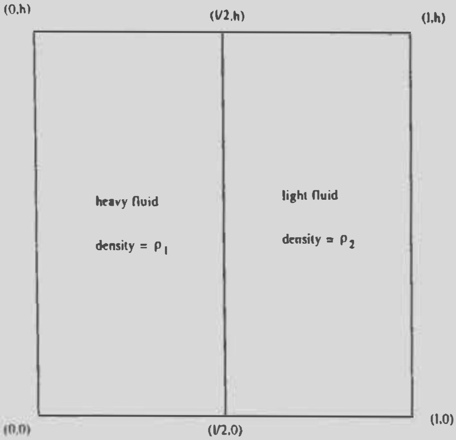



**Figure 2.** *The flow domain at $t = 0$.*

## Boundary Conditions

There is no flow through the boundaries. Hence $v_x$ and $v_y$ must be zero. Then (16) and (17) give

$$\frac{\partial P}{\partial x}(0, y) = \frac{\partial P}{\partial x}(l, y) = 0, \tag{23}$$

and

$$\frac{\partial P}{\partial y}(x, 0) = \frac{\partial P}{\partial y}(x, h) = -\rho g. \tag{24}$$

# Results

## Algorithm

1. (11), (16), and (17) are discretized using a finite difference method on a rectangular domain.

2. Using the initial values of $P$, $\rho$, $C$, and $\mu$, (11) is solved using the Successive Over Relaxation (SOR) method. SOR method approximates the solution of a system of partial differential equations. An introduction of SOR method can be found in (Peaceman, 1955).

3. Using the new pressure from step (2), $Vx$ and $Vy$ are calculated from (16) and (17), respectively.

4. The new position of a point $(x, y)$ on the front after time $\Delta t$ is $(x+v_x\Delta t, y+v_y\Delta t)$.

5. Step (4) gives the new front after time $\Delta t$. We assume that the change in the shape of the front affects the density at the neighboring points since the fluids are carried along by it. A new front after a certain time $\Delta t$ is shown in Figure 3. Initially the density at $(x, y+\Delta y)$ is $\rho_1$ and at $(x + \Delta x, y+\Delta y)$ is $\rho_2$. After time $\Delta t$ the density at the point $(x, y+\Delta y)$ is interpolated by using the formula

$$\rho(x, y + \Delta y) = \frac{\rho_1 A_1 + \rho_2 A_2}{A_1 + A_2} \tag{25}$$

where $A_1$ = area $ABDE$, and $A_2$ = area $BCD$.

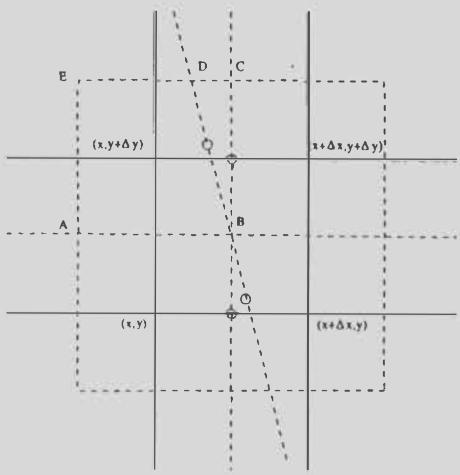



**Figure 3.** *Initial front and the front after time $t$.*

If the density at a certain grid point in the domain is $\rho$, then the average density of the fluid around the grid point in a rectangular box with length $\Delta x$ and height $\Delta y$ is $\rho$. Initially the front is vertical and the average density of the fluid in the box ABCE is $\rho_1$. But after a certain time the front divides the box ABCE into two parts, the region ABDE and the region BCD, where region BCD contains a fluid of density $\rho_2$. This affects the average density of the fluid inside the box if it is approximated using (25). It is clear that when the moving front passes completely through box ABCE (Fig. 3), the whole box is filled with the fluid of density $\rho_2$ so the average density of fluid in the box is $\rho_2$. Also (25) shows that as soon as the front passes the box ABCE, the area $A_1$ in the equation becomes zero, that is, the average density of the fluid in the box is $\rho_2$.

6. Equation (9) gives the new concentration at the grid points.

7. Using the new $P$, $\rho$, and $C$, perform steps (2)-(6) for the required number of time steps.

## Discussion

Consider a rectangular flow domain. In the vicinity of the bottom of the front, the pressure difference is higher than at any other horizontal level in the box. So the bottom of the front moves faster than any other height toward the light fluid. In a box filled with uniform sand (Fig. 4), the left side of the sand is saturated with a fluid of density $\rho_1$ and the right side is saturated with a fluid of density $\rho_2$. The front is initially vertical and in the middle of the box. Thus the pressure difference is $(\rho_1 - \rho_2)gh$. Because $g$, $\rho_1$ and $\rho_2$ are fixed, the pressure difference depends on the depth of the points only. Since the bottom of the box is the level of maximum depth, the heavy fluid moves toward the light fluid faster than any other level because velocity is proportional to pressure difference between two points. When the heavy fluid moves toward the light fluid its upper level descends and to fill the gap, the light fluid of the upper level moves toward the heavy fluid. In the bottom half of the box, the fluid particles of the heavy fluid move toward the light fluid with a velocity proportional to the depth of the point. This makes the light fluid move toward the heavy fluid in the upper half of the box. Thus, we get the moving front between the two fluids as a consequence of a general circulatory motion of the fluid. Figure 5 shows how the light fluid overrides the heavy fluid under gravitational effects.

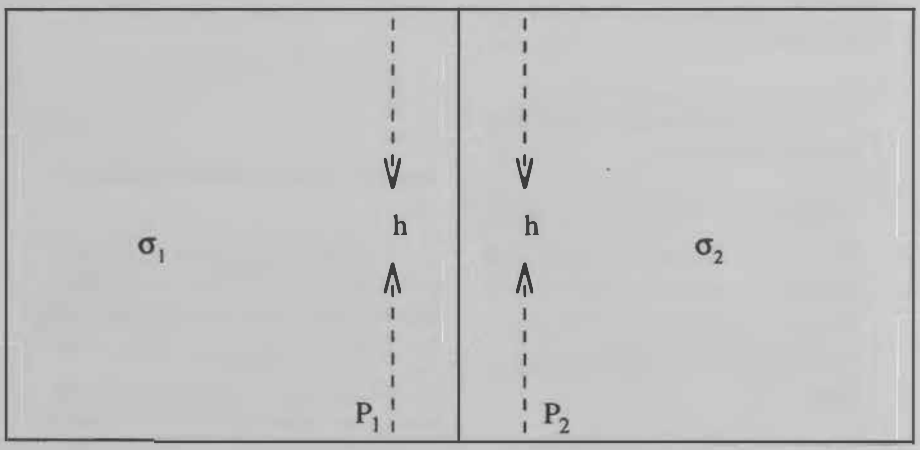
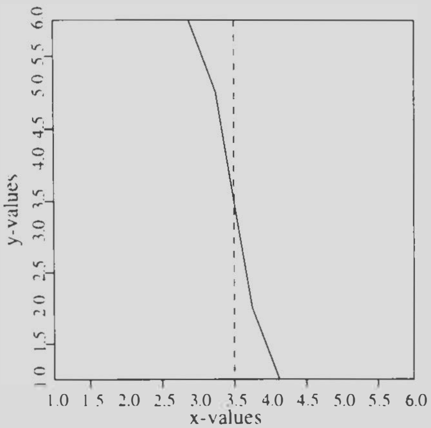
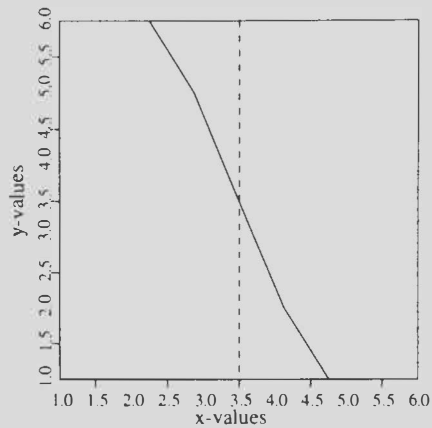


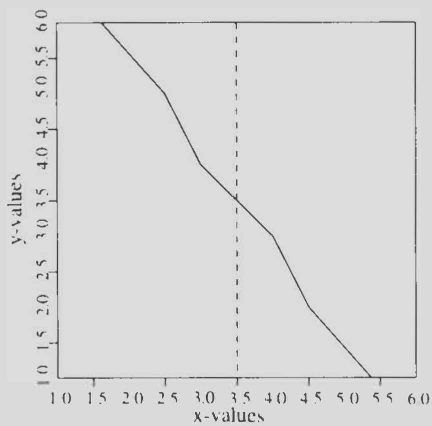

**Figure 4:** *The flow domain.*



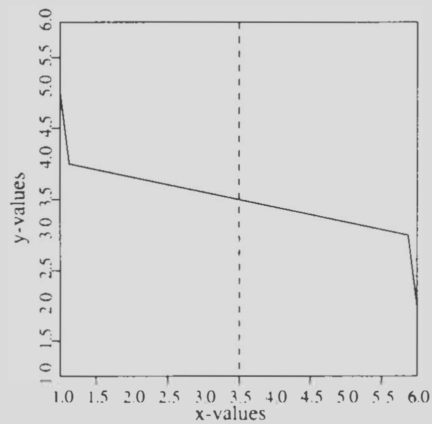

Position of the front after 1000 seconds




Position of the front after 2000 seconds




Position of the front after 4000 seconds




Position of the front after 200000 seconds

**Figure 5.** *The sharp front at different times.*

Equation (11) is solved numerically for the pressure distribution inside the domain with the initial values of $\rho_1 = 1.2$ gm/cm³, $\rho_2 = .8$ gm/cm³, $\mu_1 = \mu_2 =$ 1 centi poise, $k = 1.0$ Darcy $\left(\frac{1(cm^3/s)\ 1(centi\ poise)}{1(cm^2)\ 1(atm/cm)}\right)$ and $\phi = 0.3$.

The time step is 10 seconds. After every time step a new front is produced with a changed shape. The position of the front after 1000 seconds, 2000 seconds, 4000 seconds, and 20000 seconds is shown in Figure 5. All the graphs show the changing position of the front from its initial vertical position to a new horizontal state. Here, miscibility is considered to be zero and the sharp front between the fluids has been tracked from its initial state to its steady state. The approximate motion of the front is consistent with the analytic description of the override of the light fluid over the heavy fluid described previously.

## Conclusion

The mathematical model derived in this paper simulates the simultaneous motion of two fluids in porous media. The numerical results are consistent with the analytic description of the physical phenomena discussed in the introduction. Also the qualitative behavior of the solution is verified by an experiment done by the Petroleum Recovery Research Center at New Mexico Tech. An extension of this work would be to include the thermal effects on the flow and to consider multiphase flow instead of a single phase flow as considered in this paper. A variation of this model may be used to describe the under ground motion of a spilled contaminant that is moving towards a lake or a river.

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