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Liu Pyramids and Feedback Iteration Functions on Monoids

Abstract

General concepts of n-dimensional Liu pyramids on monoids and their feedback iteration functions are first created and established. Second, general concepts of initial sets and initial conditions are defined in order to connect the Liu pyramids and their feedback iteration functions. Four examples of 1-dimensional Liu pyramids and their feedback iteration functions, initial set, initial conditions are demonstrated. Then, the 2-dimensional case is discussed in detail. Finally, Liu notation is introduced and employed in the discussion of 3- and higher-dimensional cases.

Key words: Liu pyramid, feedback iteration function, monoid, triangle, sequence.
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Introduction

In this paper, general definitions for n-dimensional Liu pyramids and their feedback iteration functions on monoids are given in section 1. The relation between them is discussed, namely, given an initial set and a set of initial conditions on it, we will find the Liu pyramid from its feedback iteration functions and vice versa. In section 3, four examples of 1-dimensional pyramids and their feedback iteration functions are demonstrated. In section 4, several special cases of 2-dimensional pyramids on monoids, their subpyramids, and their feedback iteration functions are studied in detail. A proposed problem in the Spring 1994 issue of Math Horizons (published by M.A.A., p. 13, Problem 6) can be answered by Note (4.11) in that section. In section 5, Liu coefficient notation is introduced. Then 3- and higher-dimensional pyramids on monoids, their subpyramids, feedback iteration functions, initial set, initial conditions are discussed.

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Definitions and Notations

Let M be a nonempty abstract space. Let W be the whole number system, i.e., $W = \{0,1,2,\ldots\}$ with the usual addition and multiplication so that for a and b in W, $a - b$ is well defined if and only if a is not less than b (the trichotomy property of the whole numbers that do not form a ring). Let n be a natural number. Let W^n be the n-multiple product space of W. A mapping

$$Z = \{Z_{x_1,\ldots,x_n}\}_{(x_1,\ldots,x_n) \in W^n}$$

from W^n to M is called a n-dimensional Liu pyramid (in honor of Ju-Hsieh Liu [about 1100 A.D.]: refer to Needham and Wang 1959, Temple and Needham 1989) on M. Let $I \neq \phi$ be a subset of W^n. Let m be a natural number such that $m \leq |I|$ where $|I|$ is the cardinal number of I. Let M^m be the m-multiple product space of M. Let

$$F = \{F_X\}_{X \in W^n}$$

be a set of functions from M^m to M such that $F_X = C_X$ is a constant function for each $X \in I$. We say that Z is a Liu pyramid generated from F with the initial conditions

$$\{F_X = C_X\}_{X \in I}$$

on I provided

$$Z_X = \begin{cases} F_X & \text{if } X \in I; \\ F_X(Z_{X_1}, \cdots, Z_{X_m}) & \text{otherwise,} \end{cases}$$

where

$$Z_{X_1}, \cdots, Z_{X_m}$$

are predefined or pregenerated elements in Z. The set of functions F is called a set of feedback iteration functions for Z, I is called the initial set on W^n, and

$$\{F_X = C_X\}_{X \in I}$$

is called the set of initial conditions.

Results

1-Dimensional Liu Pyramids: Sequences

When $n = 1$, a 1-dimensional Liu pyramid is usually called a sequence. A few examples are given below:

Example (3.1). Let M be a ring with the usual addition and multiplication notations. Let e be the multiplication identity of M. Let $I = \{0\}$. Let $m = 1$. Define F as follows:

$$F_X(\alpha) = \begin{cases} e & \text{if } X \in I; \\ (Xe)\alpha & \text{otherwise.} \end{cases}$$

Let

$$Z_X = \begin{cases} F_X & \text{if } X \in I; \\ F_X(Z_{X-1}) & \text{otherwise.} \end{cases}$$

Then we obtain a factorial sequence in the ring.

Example (3.2). Let M be the whole number system with the usual addition. Let $I = \{0,1\}$. Let $m = 2$. Let F be defined as below:

$$F_X(\alpha, \beta) = \begin{cases} X & \text{if } X \in I; \\ \alpha + \beta & \text{otherwise.} \end{cases}$$

Let

$$Z_X = \begin{cases} F_X & \text{if } X \in I; \\ F_X(Z_{X-2}, Z_{X-1}) & \text{otherwise.} \end{cases}$$

It yields the Fibonacci number sequence.

Example (3.3). Let M be the real field. Let $I = \{0\}$. Let $m = 1$. Let

$$F_X(\alpha) = \begin{cases} X & \text{if } X \in I; \\ \alpha + X^{-4} & \text{otherwise} \end{cases}$$

and

$$Z_X = \begin{cases} F_X & \text{if } X \in I; \\ F_X(Z_{X-1}) & \text{otherwise.} \end{cases}$$

This is the sequence used by Euler (Peitgen and Jürgens and Saupe 1992, Temple and Needham 1989) to estimate $\pi^4/90$.

Example (3.4). Let M be the complex field. Let c be a complex number. Let $I = \{0\}$. Let $m = 1$. Let

$$F_X(\alpha) = \begin{cases} X & \text{if } X \in I; \\ \alpha^2 + c & \text{otherwise} \end{cases}$$

and

$$Z_X = \begin{cases} F_X & \text{if } X \in I; \\ F_X(Z_{X-1}) & \text{otherwise.} \end{cases}$$

This is the sequence whose boundedness is used to determine whether the complex number c is in the Mandelbrot set or not.

2-Dimensional Liu Pyramids: Triangles

When $n = 2$, the 2-dimensional Liu pyramids are usually called triangles. Let's consider some special cases:

Definition (4.1). Let $(M,\#)$ be a monoid. Let notations be the same as in section 2. Let

$$C = \{C_k\}_{k=0}^{\infty}$$

be a sequence on the monoid. Let $I = \{(k,l) \mid kl = 0 \text{ and } k,l \in W\}$. Let $m = 2$. Let

$$F_{k,l}(\alpha,\beta) = \begin{cases} C_{2l} & \text{if } k = 0; \\ C_{2k-1} & \text{if } k > l = 0, \\ \alpha\#\beta & \text{if } kl \neq 0. \end{cases}$$

Let

$$Z_{k,l} = \begin{cases} F_{k,l} & \text{if } (k,l) \in I; \\ F_{k,l}(Z_{k,l-1}, Z_{k-1,l}) & \text{otherwise.} \end{cases}$$

Then the triangle Z generated by F is called the Liu triangle on the monoid M with respect to C.

Theorem (4.2). Let notations be the same as in (4.1). Let a be an element in M. Let

$$C = \{C_k = a\}_{k=0}^{\infty}$$

Then we have

$$Z_{k,l} = \binom{k+l}{k}a = \binom{k+l}{l}a. \quad (\dagger)$$

Proof. Let Z be as defined as in $(\dagger)$. For $kl = 0$, we have

$$\binom{k+l}{k} = \begin{cases} \binom{k}{k} = 1 & \text{if } l=0; \\ \binom{l}{0} = 1 & \text{if } k=0. \end{cases}$$

So $Z_{k,l} = a = F_{k,l}$ and Z satisfies the initial conditions. Now for $kl \neq 0$, we have

$$\begin{aligned} Z_{k,l} &= \binom{k+l}{l}a \\ &= \left(\binom{k+l-1}{l-1} + \binom{k-1+l}{l}\right)a \\ &= \binom{k+l-1}{l-1}a \,\#\, \binom{k-1+l}{l}a \\ &= F_{k,l}(Z_{k,l-1}, Z_{k-1,l}). \end{aligned}$$

Thus Z satisfies the feedback iteration functions F. Therefore Z is the triangle generated from F, and the proof is complete.

Note that if $k + l = i$ is fixed, then we obtain the elements of the ith row of the triangle, namely,

$$\{Z_{k,l}\}_{k+l=i} = \{\binom{i}{l}a\}_{l=0}^{i}.$$

Example (4.3). When M is the natural number system N and $a = 1$ so that

$$C = \{C_k = 1\}_{k=0}^{\infty},$$

we get the original Liu triangle (Needham and Wang 1959, Peitgen, Jürgens, and Saupe 1992, Temple and Needham 1989).

Example (4.4). Consider the sequence

$$\{Z_{k,k} = \binom{2k}{k}\}_{k=0}^{\infty}$$

of all those elements on the bisector of the original Liu triangle. Obviously, this sequence can be constructed from the feedback iteration functions defined below:

$$F_k(\alpha) = \begin{cases} 1 & \text{if } k = 0; \\ \left\lfloor \dfrac{2(2k-1)\alpha}{k} \right\rfloor & \text{if } k > 0, \end{cases}$$

where $\alpha \in N$, with only one initial condition and each element is constructed from its exact predecessor.

In the rest of this paper, we will construct n-dimensional Liu pyramids from given feedback iteration functions and then conversely, find the feedback iteration functions for its lower dimensional subpyramids. Let us consider a special type of triangle on a ring in the next example.

Example (4.5). Let R be a ring with traditional addition and multiplication operators. Let e be the multiplication identity. Let a and b be elements in R. Let C be the sequence defined as follows:

$$C_k = \begin{cases} a & \text{if } k = 0; \\ b^{k/2} & \text{if } k \text{ is positive and even;} \\ e & \text{if } k \text{ is odd.} \end{cases}$$

Then we obtain a Liu triangle

$$Y = \{Y_{k,l}\}_{k=0,l=0}^{\infty,\infty}$$

with respect to C on the ring R. In other words, Y is a mapping from W ¥ W to R so that

$$Y_{k,l} = \begin{cases} C_{2l} & \text{if } k = 0; & (1) \\ C_{2k-1} & \text{if } k > l = 0; & (2) \\ Y_{k,l-1} + Y_{k-1,l} & \text{if } kl \neq 0. & (3) \end{cases}$$

From the above iteration rules, we will find an explicit expression for each entry of the triangle.

Theorem (4.6). Let notations be the same as above. Then we have

$$Y_{k,l} = \begin{cases} a & \text{if } k=l=0; \\ b^l & \text{if } l>k=0; \\ e & \text{if } k>l=0; \\ \binom{k+l-1}{l} e + \sum_{j=1}^{l} \binom{k+l-j-1}{l-j} b^j & \text{if } k>l>0. \end{cases}$$

Proof. It suffices to show that Y defined in this theorem satisfies the conditions (1), (2), and (3) in the above section. Plainly, $Y_{0,0} = a = C_0$ and $Y_{k,l} = b^l = C_{2l}$ for $l > k = 0$. This implies $Y_{k,l} = C_{2l}$ for $k = 0$, so (1) is satisfied. For $k > l = 0$, we get $Y_{k,l} = e = C_{2k-1}$, so (2) is satisfied also. To show Y satisfies (3) that

$$Y_{k,l} = Y_{k,l-1} + Y_{k-1,l}$$

for $kl \neq 0$. There are four cases:

Case 1: When $k = l = 1$, we get

$$\begin{aligned} Y_{1,0} + Y_{0,1} &= e + b \\ &= \binom{1}{1} e + \sum_{j=1}^{1} \binom{1+1-j-1}{1-j} b^j \\ &= Y_{1,1}. \end{aligned}$$

Case 2: When $k > 0$ and $l = 1$, we have $Y_{k,l-1} = Y_{k,0} = e$ and

$$\begin{aligned} Y_{k-1,l} &= \binom{k-1}{1} e + \sum_{j=1}^{1} \binom{k-1+1-j-1}{1-j} b^j \\ &= (k-1)e + b. \end{aligned}$$

These imply

$$\begin{aligned} Y_{k,l} &= \binom{k}{1} e + \sum_{j=1}^{1} \binom{k-j}{1-j} b^j \\ &= ke + b \\ &= e + ((k-1)e + b) \\ &= Y_{k,l-1} + Y_{k-1,l}. \end{aligned}$$

Case 3: When $k = 1$ and $l > 0$ we have

$$\begin{aligned} Y_{k,l-1} + Y_{k-1,l} &= \left(\binom{l-1}{l-1} e + \sum_{j=1}^{l-1} \binom{l-1-j}{l-1-j} b^j\right) + b^l \\ &= e + \sum_{j=1}^{l-1} b^j + b^l \\ &= e + \sum_{j=1}^{l} b^j \\ &= \binom{l}{l} e + \sum_{j=1}^{l} \binom{l-j}{l-j} b^j \\ &= Y_{k,l}. \end{aligned}$$

Case 4: When $k > 1$ and $l > 1$, we obtain

$$\begin{aligned} Y_{k,l-1} + Y_{k-1,l} &= \binom{k+l-2}{l-1}e + \binom{k+l-2}{l}e + \\ &\quad \sum_{j=1}^{l-1}\binom{k+l-2-j}{l-1-j}b^j + \sum_{j=1}^{l}\binom{k+l-2-j}{l-j}b^j \\ &= \binom{k+l-1}{l}e + \\ &\quad \sum_{j=1}^{l-1}\left(\binom{k+l-2-j}{l-1-j} + \binom{k+l-2-j}{l-j}\right)b^j + b^l \\ &= \binom{k+l-1}{l}e + \sum_{j=1}^{l-1}\binom{k+l-1-j}{l-j}b^j + b^l \\ &= \binom{k+l-1}{l}e + \sum_{j=1}^{l}\binom{k+l-1-j}{l-j}b^j \\ &= Y_{k,l}. \end{aligned}$$

Thus Y satisfies the feedback iteration functions in (3) and the proof is completed.

Now consider the sequence of those points $Y_{k,l}$ in Y such that $k - l - 1 = 0$. They form a subtriangle of the triangle. We will find its feedback iteration functions. Let us state the next theorem first.

Theorem (4.7). Let notations be the same as in (4.6). Then we have

$$Y_{k+1,k} = \begin{cases} e & \text{if } k = 0; \\ \binom{2k+1}{k}e & \text{if } b = e; \\ f_k(Y_{k,k-1}) & \text{if } k > 0 \text{ and } b-e \text{ is invertible,} \end{cases}$$

where

$$f_k(X) = (b-e)^{-1}\left(\binom{2k+1}{k}(b-2e) + b^2X\right).$$

Proof. If $k = 0$, from theorem (4.6), we get $Y_{k+1,k} = Y_{1,0} = e$. Now if $k > 2$, again making use of theorem (4.6), we then have

$$\begin{aligned} Y_{k+1,k} &= \binom{2k}{k}e + \sum_{j=1}^{k}\binom{2k-j}{k-j}b^j \\ &= \binom{2k}{k}e + \sum_{j=1}^{k-1}\binom{2k-j}{k-j}b^j + \binom{k}{0}b^k \\ &= \binom{2k-1}{k}e + \binom{2k-1}{k-1}e + \sum_{j=1}^{k-1}\binom{2k-j-1}{k-j-1}b^j \\ &\quad + \sum_{j=1}^{k-1}\binom{2k-j-1}{k-j}b^j + \binom{k-1}{0}b^k. \end{aligned}$$

This yields, by setting $j+1=h$ and $j-1=i$,

$$\begin{aligned} bY_{k+1,k} &= \binom{2k-1}{k}b + \binom{2k-1}{k-1}b + \sum_{j=1}^{k-1}\binom{2k-j-1}{k-j-1}b^{j+1} \\ &\quad + b^2\left(\binom{2k-2}{k-1}e + \sum_{j=2}^{k-1}\binom{2k-j-1}{k-j}b^{j-1} + \binom{k-1}{0}b^{k-1}\right) \\ &= \binom{2k-1}{k}b + \left(\binom{2k}{k}e + \binom{2k-1}{k-1}b + \sum_{h=2}^{k}\binom{2k-h}{k-h}b^h - \binom{2k}{k}e\right) \\ &\quad + b^2\left(\binom{2k-2}{k-1}e + \sum_{i=1}^{k-2}\binom{2k-2-i}{k-1-i}b^i + \binom{k-1}{0}b^{k-1}\right) \\ &= \binom{2k-1}{k}b + \left(Y_{k+1,k} - \binom{2k}{k}e\right) + b^2Y_{k,k-1}. \end{aligned}$$

It implies

$$\begin{aligned} (b-e)\,Y_{k+1,k} &= \binom{2k-1}{k}b - \binom{2k}{k}e + b^2Y_{k,k-1} \\ &= \binom{2k-1}{k}b - 2\binom{2k-1}{k}e + b^2Y_{k,k-1} \\ &= \binom{2k-1}{k}(b-2e) + b^2Y_{k,k-1}. \end{aligned} \qquad (*)$$

One may check that (*) holds for $1 \le k \le 2$ also. Now if $b=e$, then we have

$$0 = \binom{2k-1}{k}(-e) + e^2Y_{k,k-1}.$$

So

$$Y_{k,k-1} = \binom{2k-1}{k}e = \binom{2k-1}{k-1}e,$$

for $k>0$. Thus we obtain

$$Y_{k+1,k} = \binom{2k+1}{k}e,$$

for $k \ge 0$.

Again from (*), if $k>0$ and $b-e$ is invertible, then we obtain

$$Y_{k+1,k} = (b-e)^{-1}\left(\binom{2k-1}{k}(b-2e) + b^2Y_{k,k-1}\right).$$

This completes the proof.
From the above theorem, we see that the sequence

$$\{Y_{k+1,k}\}_{k=0}^{\infty}$$

is generated by a set of feedback iteration functions. We state it formally.

Corollary (4.8). Let notations be the same as in (4.5). If $b-e$ is invertible, then the sequence

$$\{Y_{k+1,k}\}_{k=0}^{\infty}$$

is generated by the feedback iteration functions defined below:

$$f_k(x) = \begin{cases} e & \text{if } k=0; \\ (b-e)^{-1}\left(\binom{2k+1}{k}(b-2e)+b^2x\right) & \text{if } k>0 \end{cases}$$

for $x \in R$.

Next let us consider the special case when $b=2e$.

Corollary (4.9). Let notations be the same as in (4.5). If $b=2e$, then the sequence

$$\{Y_{k+1,k}\}_{k=0}^{\infty}$$

is generated by the feedback iteration functions defined below:

$$f_k(x) = \begin{cases} e & \text{if } k = 0; \\ 4x & \text{if } k > 0 \end{cases}$$

for $x \in R$.

Proof. Since $b = 2e$, we have $(b - e)^{-1} = e$, the first term in the parentheses vanishes, and the second term becomes

$$(2e)^2 x = 4x.$$

This completes the proof.

By mathematical induction, the proof of the next corollary is straight forward by Theorem (4.7) and Corollary (4.9).

Corollary (4.10). Let notations be the same as in (4.5). If $b = 2e$, then we have

$$Y_{k+1,k} = 4^k e$$

for all whole numbers k.

Now let us consider a special case below:

Note (4.11). In (4.5), if we select the ring to be the integer ring and set $a = 1$ and $b = 2$, then we obtain the triangle shown in Math Horizons (1994: 13). A solution to that problem 6 is given by the specia- case of the above corollary when $e = 1$ in the integer ring. The original problem is rewritten as follows:

Problem 6. In the variation on Pascal's Triangle, the 1's on the right were replaced by successive powers (from 0 up) of 2. Show that the numbers in the column on the left of the symmetric axis of the triangle are the successive powers of 4 (also from 0 up).

Three and Higher Dimensional Liu Pyramids

Before we proceed, let us introduce the Liu coefficient or notation and show some of its important properties.

Notation (5.1). Let $x_1,\ldots,x_n$ be whole numbers. Define

$$\langle x_1, \cdots, x_n \rangle = \frac{(x_1 + \cdots + x_n)!}{x_1! \cdots x_n!}$$

This is called the Liu coefficient notation. Note that when $n = 2$, we have

$$\langle x_1, x_2 \rangle = \binom{x_1 + x_2}{x_1} = \binom{x_1 + x_2}{x_2}.$$

Lemma (5.2). For every natural number n, we have

$$\langle x_1, \cdots, x_n \rangle = \sum_{j=1}^{n} \langle x_1, \cdots, x_j - 1, \cdots, x_n \rangle$$

provided that $x_j > 0$ for all $1 \le j \le n$.

Proof. One may check that the lemma is true for $n = 2$. Assume that the lemma is true for $2 \le n \le m$. Then

$$
\begin{aligned}
\langle x_1,\cdots,x_{m+1}\rangle &= \frac{(x_1+\cdots+x_{m+1})!}{x_1!\cdots x_{m+1}!}\\
&= \frac{(x_1+\cdots+x_m)!}{x_1!\cdots x_m!}\;\frac{((x_1+\cdots+x_m)+x_{m+1})!}{(x_1+\cdots+x_m)!\,x_{m+1}!}\\
&= \frac{(x_1+\cdots+x_m)!}{x_1!\cdots x_m!}\Big(\frac{(x_1+\cdots+x_{m+1}-1)!}{(x_1+\cdots+x_m-1)!\,x_{m+1}!}\\
&\quad+\frac{(x_1+\cdots+x_{m+1}-1)!}{(x_1+\cdots+x_m)!\,(x_{m+1}-1)!}\Big)\\
&= \frac{(x_1+\cdots+x_m)!}{x_1!\cdots x_m!}\;\frac{(x_1+\cdots+x_{m+1}-1)!}{(x_1+\cdots+x_m-1)!\,x_{m+1}!}\\
&\quad+\frac{(x_1+\cdots+x_{m+1}-1)!}{x_1!\cdots x_m!\,(x_{m+1}-1)!}\\
&= \Big(\sum_{j=1}^{m}\langle x_1,\cdots,x_j-1,\cdots,x_m\rangle\Big)\\
&\quad\frac{(x_1+\cdots+x_{m+1}-1)!}{(x_1+\cdots+x_m-1)!\,x_{m+1}!}+\frac{(x_1+\cdots+x_{m+1}-1)!}{x_1!\cdots x_m!\,(x_{m+1}-1)!}\\
&= \Big(\sum_{j=1}^{m}\frac{(x_1+\cdots+x_m-1)!}{x_1!\cdots(x_j-1)!\cdots x_m!}\Big)\frac{(x_1+\cdots+x_{m+1}-1)!}{(x_1+\cdots+x_m-1)!\,x_{m+1}!}\\
&\quad+\frac{(x_1+\cdots+x_{m+1}-1)!}{x_1!\cdots x_m!\,(x_{m+1}-1)!}\\
&= \sum_{j=1}^{m+1}\langle x_1,\cdots,x_j-1,\cdots,x_{m+1}\rangle.
\end{aligned}
$$

This implies that the lemma is true for $n = m + 1$. Therefore, by mathematical induction, the lemma is proved.

Theorem (5.3). Let $(x_1,\ldots,x_n) \neq 0$. Then we have

$$\langle x_1,\cdots,x_n\rangle = \sum_{x_j>0}\langle x_1,\cdots,x_j-1,\cdots,x_n\rangle.$$

Proof. Since Liu notation is independent of x_j's order, without loss of generality, we may assume that $x_j > 0$ for $1 \leq j \leq h < n$ and $x_j = 0$ for $h < j \leq n$. Then, by the above lemma, we have

$$
\begin{aligned}
\langle x_1,\cdots,x_n\rangle &= \langle x_1,\cdots,x_h\rangle\\
&= \sum_{j=1}^{h}\langle x_1,\cdots,x_j-1,\cdots,x_h\rangle\\
&= \sum_{j=1}^{h}\langle x_1,\cdots,x_j-1,\cdots,x_n\rangle\\
&= \sum_{x_j>0}\langle x_1,\cdots,x_j-1,\cdots,x_n\rangle.
\end{aligned}
$$

Now, given a monoid, we are ready to define a special type of n-dimensional pyramid on it:

Definitions and Notations (5.4). Let $(M,\#)$ be a monoid. Let

$$I = \{(x_1,\cdots,x_n) \mid \sum_{j=i} x_j^2 = 0 \text{ for some } 1 \leq i \leq n\}$$

be a subset of W^n. I is called the set of edges of the n-dimensional pyramid. Let

$$C = \{C_{x_1,\cdots,x_n}\}_{(x_1,\cdots,x_n)\in I}$$

be a mapping from I to M. Let

$$
F_{x_1,\cdots,x_n}(\alpha_1,\cdots,\alpha_n) =
\begin{cases}
C_{x_1,\cdots,x_n} & \text{if } (x_1,\cdots,x_n)\in I;\\
\#_{j=1}^{n}\, sign(x_j)\,\alpha_j & \text{otherwise,}
\end{cases}
$$

where $(a_1,\ldots,a_n)$ Œ M^n. Let

$$Z_{x_1,\ldots,x_n} = \begin{cases} F_{x_1,\ldots,x_n} & \text{if } (x_1,\cdots,x_n) \in I; \\ F_{x_1,\ldots,x_n}(Z_{\max(x_1-1,0),\ldots,x_n},\cdots,Z_{x_1,\ldots,\max(x_j-1,0),\ldots,x_n},\cdots, \\ \quad Z_{x_1,\ldots,\max(x_n-1,0)}) & \text{otherwise.} \end{cases}$$

We call Z the n-dimensional Liu pyramid with respect to C. In the next theorem, we will show an explicit expression for each entry of the special pyramid when all the initial conditions have the same value.

Theorem (5.5). Let notations be the same as in (5.4). Let a be in M. Let

$$C_{x_1,\ldots,x_n} = a$$

for all $(x_1,\ldots,x_n)$ in I. Let Z be the pyramid with respect to C. Then, for all $(x_1,\ldots,x_n)$ in W^n, we have

$$Z_{x_1,\ldots,x_n} = \langle x_1,\cdots,x_n\rangle a$$

Proof. Let Z be defined as in the theorem. It suffices to show that it satisfies the initial conditions and feedback iteration functions in (5.4). For $(x_1,\ldots,x_n) \in I$, we get

$$\begin{aligned} Z_{x_1,\ldots,x_n} &= \langle x_1,\cdots,x_n\rangle a \\ &= a \\ &= C_{x_1,\ldots,x_n} \\ &= F_{x_1,\ldots,x_n}. \end{aligned}$$

Thus, the initial conditions are satisfied. For $(x_1,\ldots,x_n) \notin I$, by (5.3), we have

$$\begin{aligned} & F_{x_1,\ldots,x_n}(Z_{\max(x_1-1,0),\ldots,x_n},\cdots, \\ & \qquad Z_{x_1,\ldots,\max(x_j-1,0),\ldots,x_n},\cdots, \\ & \qquad Z_{x_1,\ldots,\max(x_n-1,0)}) \\ &= \#_{j=1}^{n} \operatorname{sign}(x_j)\, Z_{x_1,\ldots,\max(x_j-1,0),\ldots,x_n} \\ &= \#_{x_j>0} Z_{x_1,\ldots,\max(x_j-1,0),\ldots,x_n} \\ &= \#_{x_j>0} \langle x_1,\cdots,x_j-1,\cdots,x_n\rangle a \\ &= \Big(\sum_{x_j>0} \langle x_1,\cdots,x_j-1,\cdots,x_n\rangle\Big) a \\ &= \langle x_1,\cdots,x_n\rangle a \\ &= Z_{x_1,\ldots,x_n}. \end{aligned}$$

Thus, the feedback iteration functions are satisfied also. This completes the proof.

For the case when $n = 3$, let us consider two of its special subpyramids, namely,

$$S = \{S_k = Z_{k,k,k}\}_{k=0}^{\infty}$$

and

$$T = \{T_{k,l} = Z_{k,l,l}\}_{k=0,\,l=0}^{\infty,\infty}.$$

We want to find their feedback iteration functions.

Corollary (5.6). Let S and T be the same as in the discussion after (5.5). We then have

(1) S is the sequence on M generated by

$$F_k(\alpha) = \begin{cases} a & \text{if } k = 0; \\ \dfrac{(3k)!}{(k!)^3} a & \text{if } k > 0 \end{cases}$$

so that $S_k = F_k$. Note that all F_k are constant mappings from M to itself.

(2) T is a triangle generated by

$$F_{k,l}(\alpha,\beta) = \begin{cases} (k+1)a & \text{if } l = 0; \\ (l+1)a & \text{if } k = 0; \\ \alpha\#\beta\#\binom{k+l}{l}a & \text{if } kl \neq 0 \end{cases}$$

so that

$$T_{k,l} = \begin{cases} F_{k,l} & \text{if } kl = 0; \\ F_{k,l}(T_{k,l-1}, T_{k,l-1}) & \text{otherwise.} \end{cases}$$

Proof. This corollary is a direct consequence of the definitions of S_k and

$T_{k,l}$ after (5.5) and the following identities:

$$Z_{k,k,k} = \frac{(3k)!}{(k!)^3}a$$

and

$$Z_{k,l,1} = \begin{cases} Z_{k,0,1} & \text{if } l = 0; \\ Z_{0,l,1} & \text{if } k = 0; \\ Z_{k,l-1,1} \# Z_{k-1,l,1} \# Z_{k,l,0} & \text{if } kl \neq 0, \end{cases}$$

where

$$Z_{k,0,1} = (k+1)a,$$

$$Z_{0,l,1} = (l+1)a, \text{ and}$$

$$Z_{k,l,0} = \binom{k+l}{k}a$$

To close this section, we give an alternative definition for the Liu notation by a single initial condition on the vertex of the pyramid and a simple iteration rule.

Note (5.7). When M and a are respectively replaced by the integer ring and 1, we find that Liu coefficients form the Liu pyramid with initial value 1 on its edges. Note also that the Liu notation pyramid can also be defined directly as follows:

$$Z_{x_1,\cdots,x_n} = \begin{cases} 1 & \text{if } (x_1,\cdots,x_n) = 0; \\ \sum_{x_j>0} Z_{x_1,\cdots,x_j-1,\cdots,x_n} & \text{otherwise.} \end{cases}$$

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