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ESSENTIALLY NON-OSCILLATORY METHODS FOR NUMERICAL WAVE PROPAGATION

ABSTRACT

In modeling applications involving wave propagation (seismic data analysis for example), one needs to obtain very accurate numerical solutions to wave equations. Standard finite difference (FD) techniques suffer serious shortcomings—numerical dispersion and spurious oscillations. The essentially non-oscillatory (ENO) method overcomes these shortcomings. It utilizes an adaptive finite difference technique and is applied to a system of first order partial differential equations that is equivalent to the acoustic wave equation. The ENO method and FD techniques are applied to an acoustic wave equation with constant wave speed in one dimension with both smooth and non-smooth initial data. These simple examples are sufficient to illustrate how spurious oscillations are introduced into the FD approximation when the initial data is not smooth, and shows that the ENO method does not suffer the same effects. A two dimensional problem is also presented with similar results.

Key words: numerical wave propagation, ENO methods

INTRODUCTION

The need for accurate, efficient numerical methods for solving wave equations arises in many applications, e.g., seismic inversion, medical ultrasound imaging, and radar and sonar imaging. The methods to be considered in this paper apply to time-dependent wave equations like the acoustic (scalar) wave equation

$$\partial_t^2 u - c^2 \left(\partial_x^2 u + \partial_y^2 u + \partial_z^2 u\right) = 0. \quad (1)$$

This equation models the propagation of a disturbance through a fluid at speed c. A simple example is sound traveling through the air. These methods can also be applied to models for more complicated phenomena like electromagnetic waves (e.g., radar) and elastic waves (e.g., shock waves traveling through the earth).

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The standard approach to numerically solving wave equations like (1) is the finite difference (FD) method. The FD method is based on approximations to derivatives obtained using Taylor expansions. It has the following features:

(i) If the solution u is smooth (i.e., if it has continuous higher order derivatives), then it yields very accurate approximations.

(ii) If the solution u is *not smooth*, spurious oscillations may arise in the approximations.

(iii) The method suffers from a phenomena known as "numerical dispersion". This means that FD approximations to sharp features have a tendency to spread, or loss sharpness, as they propagate.

The shortcomings (ii)-(iii) may have very serious ramifications. For instance, in seismic prospecting, spurious oscillations may cause one to predict structure where none exists. Numerical dispersion may cause one to miss structure that should be present.

In this paper we apply essentially

non-oscillatory (ENO) numerical methods to the scalar wave equation (1). As the name implies, these methods do not suffer from shortcoming (ii) of standard FD methods. These methods also have the desirable property of maintaining sharp wave fronts. Hence their popularity in solving model equations which describe phenomena like flame front propagation (Osher and Sethian 1988). ENO methods have also recently been applied to solve equations which arise when high frequency asymptotic methods are applied to (1) (Engquist et al. 1993).

This paper is organized as follows: The next section contains a very brief discussion of the standard FD method for the scalar wave equation (1). This is followed by a brief introduction to the ENO method and its implementation for the scalar wave equation in section 3. Some preliminary numerical results are presented in the final section.

The Finite Difference (FD) Method

FD methods can be derived from Taylor approximations. For instance, from

$$u(x_i, t_k + \Delta t) = u(x_i, t_k) +$$

$$\partial_t u(x_i, t_k)\,\Delta t + \text{Higher Order Terms},$$

one obtains the difference approximation

$$\partial_t u(x_i, t_k) \approx \frac{u(x_i, t_k + \Delta t) - u(x_i, t_k)}{\Delta t}.$$

To derive a numerical method, one lays-down a grid, or mesh, on the region of interest and approximates the solution u at the mesh points (χ_i, t_k) by the components $u_{i,k}$ of a mesh function. These components solve a discrete system obtained from the differential equation by replacing derivatives with their difference approximations.

When the FD method is applied to the one-dimensional version of the scalar wave equation (1),

$$\partial_t^2 u - c^2 \partial_x^2 u = 0, \qquad (2)$$

the discrete system takes the form

$$\frac{u_{i,k-1} - 2u_{i,k} + u_{i,k+1}}{\Delta t^2} - c^2 \frac{u_{i-1,k} - 2u_{i,k} + u_{i+1,k}}{\Delta x^2} = 0. \qquad (3)$$

One can explicitly solve for $u_{i,k+1}$ and obtain an algorithm to explicitly "time-march" from the previous time levels to the next time level, t_{k+1}. Provided certain smoothness and stability conditions are met, this method is second order accurate in both the spatial and temporal discretization levels Δx and Δt. This, together with ease of implementation, account for its popularity.

The ENO Method

In [1], Crandall and Lions adapt the ENO method for the numerical solution of equations of Hamilton-Jacobi type,

$$\partial_t u + H(\partial_x u) = 0. \qquad (4)$$

Here *H* is a (possibly nonlinear) scalar-valued function. In order to apply ENO, the differential equation (1) must first be converted to Hamilton-Jacobi form. To illustrate this conversion, consider the one-dimensional scalar wave equation (2) and assume $c = c(x)$.

We first put (2) in first order system form,

$$\partial_t \vec{v} + A\partial_x \vec{v} = \vec{0}, \qquad (5)$$

where

$$\vec{v} = \begin{bmatrix} \partial_x u \\ \partial_t u \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -1 \\ -c^2 & 0 \end{bmatrix}.$$

Next, we diagonalize the matrix A using the eigendecomposition $A = EDE^{-1}$, with

$$E = \begin{bmatrix} 1 & 1 \\ -c & c \end{bmatrix}, \qquad D = \begin{bmatrix} c & 0 \\ 0 & -c \end{bmatrix}.$$

Left multiplying (5) by E^{-1} and defining $\vec{w} = E^{-1}\vec{v}$, one obtains by the product rule

$$\partial_t \vec{w} + D\partial_x \vec{w} = D\partial_x E^{-1} E\vec{w}. \qquad (6)$$

Since

$$D\partial_x E^{-1} E = \frac{c'}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix},$$

the equation (6) has component form (with $\vec{w} = [w^{(1)}, w^{(2)}]^T$)

$$\partial_t w^{(1)} + c\partial_x w^{(1)} = \frac{c'}{2}(w^{(2)} - w^{(1)}) \qquad (7)$$

$$\partial_t w^{(2)} - c\partial_x w^{(2)} = \frac{c'}{2}(w^{(2)} - w^{(1)}). \qquad (8)$$

Here the prime (′) denotes differentiation with respect to χ. When the wave speed c is constant, $c' = 0$, and this system decouples into two Hamilton-Jacobi equations with $H = \pm c\partial_x u$. When $c' \neq 0$, the ENO method can still be applied, since the principal parts (derivative terms) are in Hamilton-Jacobi form. Once the solution to (6) has been obtained, the solution to u the original system (5) is obtained by backtransforming, $\vec{v} = E\vec{w}$.

The ENO method for system (7)-(8) uses the same computational grid as the FD technique described in the previous section. Basically, this involves using a forward time-difference approximation to the $\partial_t w^{(1)}$ and $\partial_t w^{(2)}$ terms and adaptive spatial difference approximations to the $\partial_x w^{(1)}$ and $\partial_x w^{(2)}$ terms, as described in Crandall and Lions 1984 and outlined below. We make a minor modification to handle the non-zero terms on the right hand side.

The principle part of equation (7) corresponds to propagation of information from left to right. A standard numerical approach for directional wave propagation, called "upwinding", requires (in this case) the derivative of $w^{(1)}$ with respect to x from the left. ENO can be viewed as a scheme for adaptively computing this (left) derivative approximation. At each spatial point x_i one constructs an interpolatory polynomial $P_i^{(1)}(x)$ through data $\{x_j, w^{(1)}(x_j,t)\}$, where the points x_j are "near" x_i. The number of points used determines the accuracy of the spatial discretization, and remains fixed. The construction is adaptive in the sense that the set of points used is selected to minimize the oscillation of the polynomial. Near the computational boundary, the order of the approximation can be maintained, but one limits the selection procedure to incorporate boundary data and utilize only points inside the region of interest. A rigorous formulation of the selection procedure and polynomial construction can be found in the appendix of Osher and Sethian 1988. One then takes

$$l_i = \frac{d}{dx}P_i^{(1)}(x_i) \qquad (9)$$

to approximate the left derivative of $w^{(1)}$. Then (9) and a forward difference in time substituted for $\partial_t w^{(1)}$ in (7) yield

$$w_{i,k+1}^{(1)} = w_{i,k}^{(1)} - \Delta t\, c(x_i)\, l_i + \frac{c'(x_i)}{2}(w_{i,k}^{(2)} - w_{i,k}^{(1)}).$$

A similar computation based on equation (8) with an ENO approximation to the derivative of $w^{(2)}$ with respect to x from the right is used to propagate $w^{(2)}$ from the k^{th} to the $(k+1)^{th}$ time level.

Crandall and Lions (see [1]) have shown that even when the solution is not smooth, this ENO scheme converges and its accuracy is $O(\sqrt{\Delta t})$. When the solution is smooth and n^{th} degree interpolating polynomials are used, $O(\Delta x^n) + O(\Delta t)$ accuracy can be attained.

We have extended this technique to higher dimensional scalar wave equations. For example, the two-dimensional version

$$\partial_t^2 u - c^2(\partial_x^2 u + \partial_y^2 u) = 0 \qquad (10)$$

can be written in system form

$$\partial_t \vec{v} + \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -c^2 & 0 & 0 \end{bmatrix} \partial_x \vec{v} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -c^2 & 0 \end{bmatrix} \partial_y \vec{v} = \vec{0}.$$

where $\vec{v} = [\partial_x u \; \partial_y u \; \partial_t u]^T$. We then apply an operator splitting method due to Strang (LeVeque 1992) to break the problem in two space dimensions into a series of one-dimensional problems, each of which yield systems analogous to (7)-(8). A similar approach can be applied to the three-dimensional scalar wave equation (1).

A Numerical Comparison

In this section we present numerical results comparing FD and ENO for one- and two-dimensional scalar wave equations.

When the wave speed c is constant, the solution to the one-dimensional scalar wave equation (2) with the initial data

$$u(x,0) = f(x), \quad \partial_t u(x,0) = 0 \qquad (11)$$

is

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct)). \qquad (12)$$

Example 1: A Smooth Solution in 1-D.

The first example has a solution to (2) which is smooth. Take $c = 1$ and initial data.

$$u(x,0) = f(x) = \begin{cases} 0 & x < 1.5 \\ \sin^4(\pi \frac{x-2}{2}) & 1.5 \le x \le 2.5 \\ 0 & x > 2.5 \end{cases}$$

The exact solution is obtained using (12). In order to compare the two methods, we display the computed values of $\partial_x u$ rather than u itself. The ENO method calculates this directly, while the FD method calculates u and from that, a high or order approximation to $\partial_x u$ computed. The spatial and temporal mesh spacings are

$$\Delta x = \frac{1}{50}, \quad \Delta t = \frac{\Delta x}{4}.$$

Figure 1 shows "fixed time snapshots", or plots of $\partial_x u$ as a function of x for fixed $t = t$, at times $t = 0$, .5, 1, and 1.5. Plots in the left column were obtained using FD, while those on the right were obtained from ENO. In each of these plots, the computed approximations are illdistinguishable from the exact solution.

Example 2: A Non-Smooth Solution in 1-D.

This second example involves a solution to (2) which is not smooth. Take $c = 1$ and the initial data

$$u(x,0) = f(x) = \begin{cases} 0 & x < 1.5 \\ 4x - 6 & 1.5 \le x < 1.75 \\ 1 & 1.75 \le x < 2.25 \\ -4x + 10 & 2.25 \le x < 2.5 \\ 0 & x > 2.5 \end{cases}$$

with the same grid spacings as in Example 1. Figure 2 gives fixedtime snapshots of the computed approximation to ∂_{cu} for this case. Again, plots in the left column are for the FD method, while those in the right column are for the ENO method. Here we see that as time progresses, spurious oscillations enter into the FD approximation. Note also that the oscillations spread (disperse) with increasing time throughout the entire computational domain. On the other

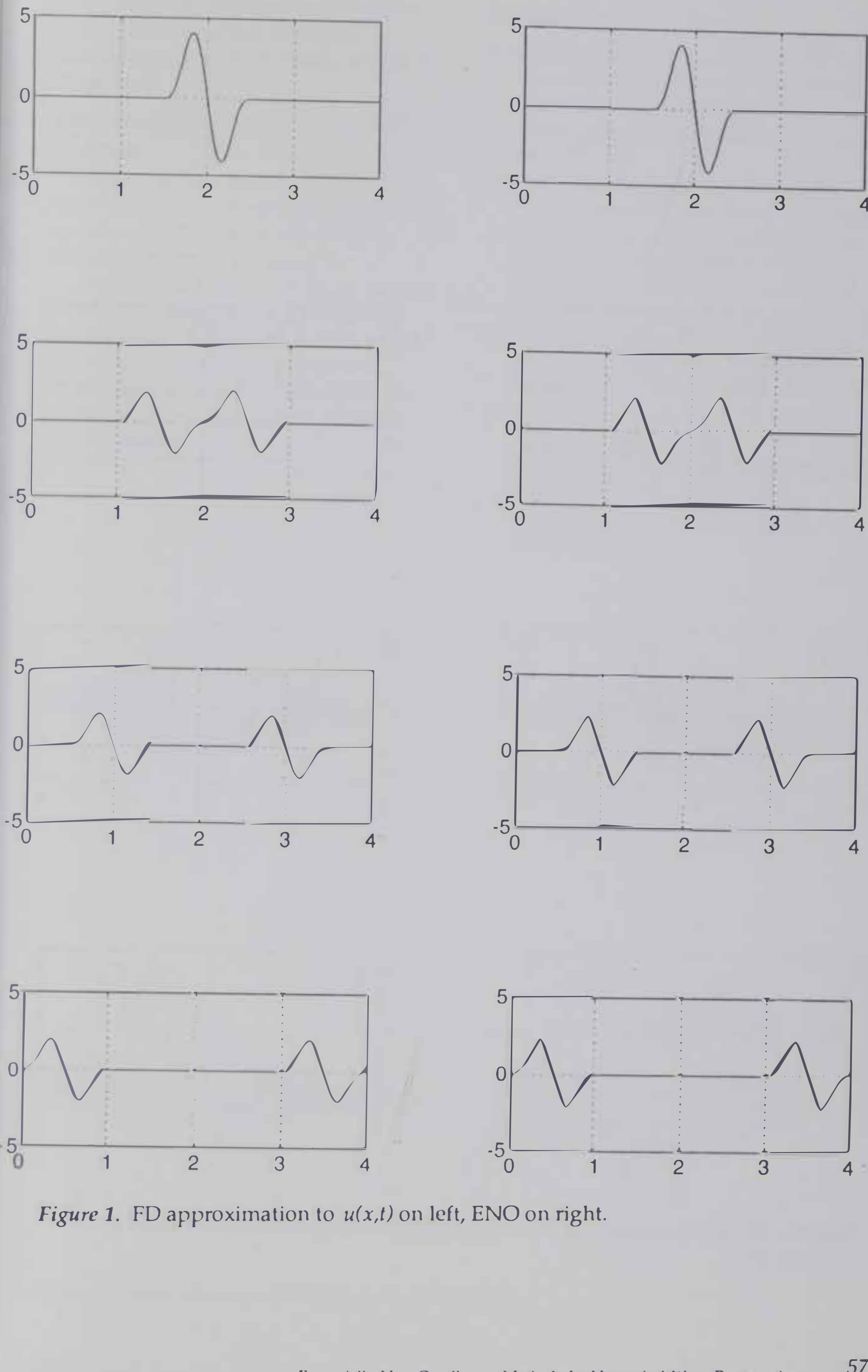

Figure 1. FD approximation to $u(x,t)$ on left, ENO on right.

Figure 2. FD approximation to $u(x,t)$ on left, ENO on right (the square wave is the exact solution).

hand, the ENO approximations in the right column have no spurious oscillations and only a small amount of dispersion.

Example 3: A Nonsmooth 2-D Solution.

Consider the two-dimensional scalar wave equation (10) with constant wave speed $c = 1$. The initial data for this example is shown in Figure 3.

In Figures 4 and 5 we again display fixed-time snapshots of $\partial_x u$. Figure 4 contains the FD approximations and 5 contains the ENO approximations. Again the FD solutions have spurious oscillations, while the ENO solutions clearly do not.

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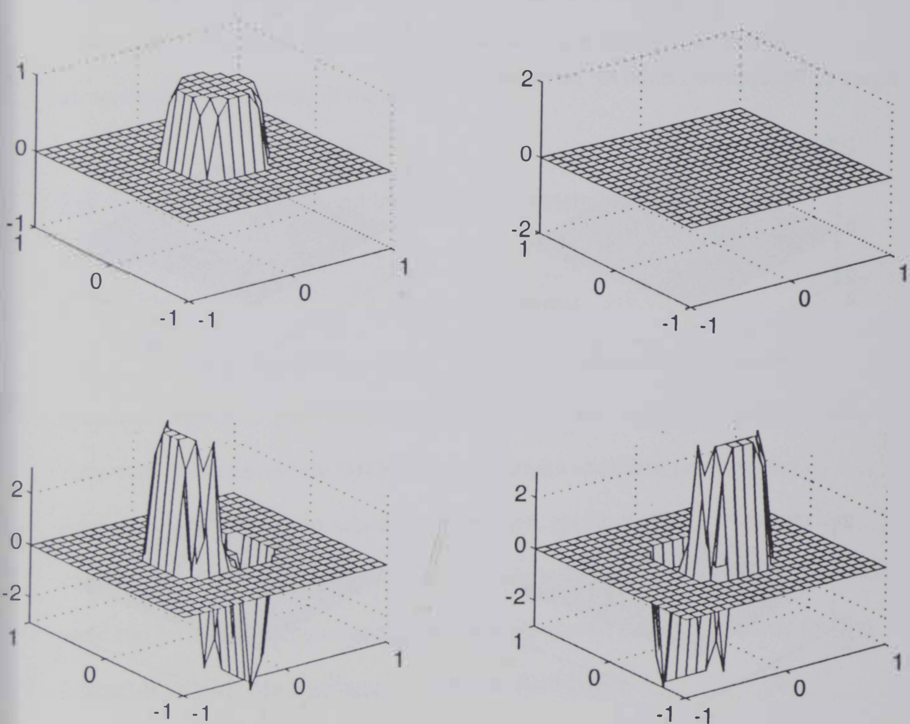

Figure 3. Initial data for two dimensional example.

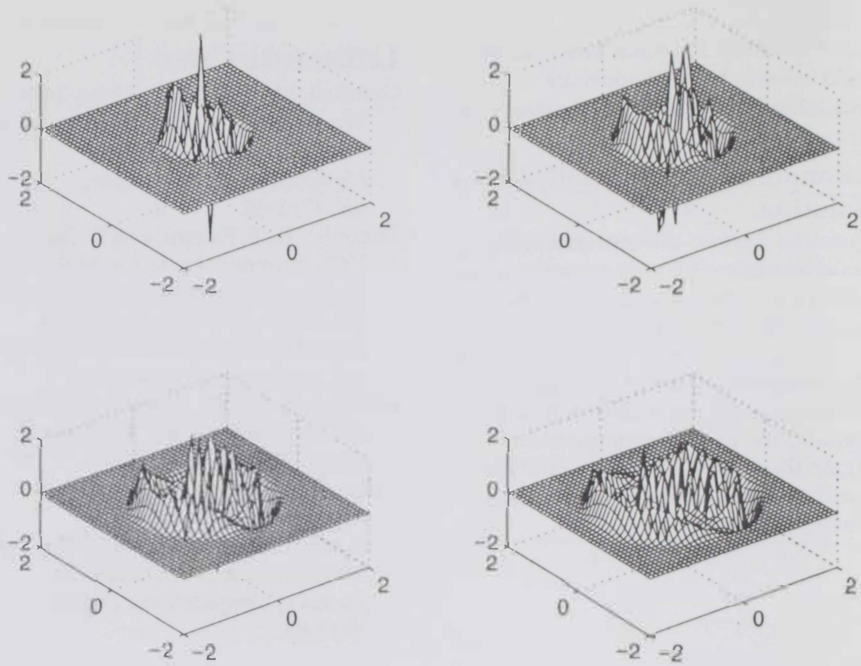

Figure 4. FD approximation to $xu\,(x,y,t)$.

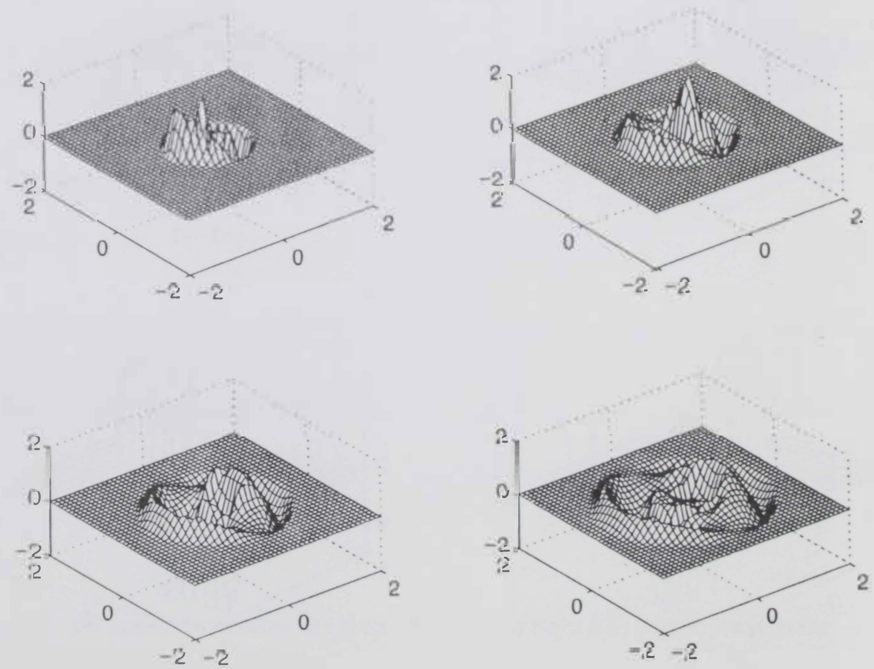

Figure 5. ENO approximation to $xu(x,y,t)$.