The Stochastic Model of Snow Cover Stability on Mountain Slopes

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ABSTRACT

Three-demensional deterministic model of thin elastic shell on a rigid underlying surface of arbitrary configuration is used as approach to a snow slab on the mountain slope. The finite difference method is used for calculation of stress distributions in the snow cover. Spatial distributions of snow cover characteristics are represented as stochastic fields which realisations are simulated with Monte Carlo method. Such characteristics as snow thickness, density and cohesion were simulated on a base of information about spatial statistical structure of these parameters to obtain a stress field over a slope.

INTRODUCTION

The forecasting of avalanche release can be made using estimates of current snowpack stress field. Such estimates may be obtained using an information of snow thickness, snow density, shear and tensile strength and dry friction coefficient. If values of these parameters are known at any point of the snowpack, one able to compute stress field by any numerical method and determine potentially dangerous zones where the stress exceeds some threshold level of stress.

Such simple scheme rarely may be applied to predict an avalanche release or to determine a dangerous zones in deterministic manner. The spatial variability of snowpack parameters is significant and can not be determined in practice with sufficient resolution. This fact stimulate the using of probabilistic methods, where the probability density and covariations of parameters will be used instead of exact values of them.

PRESENTATION OF THE PROBLEM

We study a stationary deterministic and stochastic problem of snowpack balance on arbitrary shaped mountain slope. The problem of calculating the stress field in a snowpack lying on a mountain slope of arbitrary shape is in fact a 3-dimensional problem. Being solved numerically as 3D it is a very time-consuming task. As has been shown in previous studies (Nye J.K., 1959, Nefed'ev V.O and Bozhinsky A.N., 1989) the 3D problem may be reduced to 2D if the parameters of snow depend weakly on snow depth.

The mostly appropriate coordinate system for this problem is a local orthogonal basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, where $\mathbf{e}_1$, $\mathbf{e}_2$ are the unit vectors tangential to the two curvature line at any point of surface, and $\mathbf{e}_n = \mathbf{e}_1 \mathbf{x} \mathbf{e}_2$. Here we assume that all points of considered surface are non-umbilic, hence at any point two different curvature line exist. Under such assumptions the stress field governed by the simple partial differential equations (PDE) of balance of a thin elastic non-moment shell (Novojilov V.V, 1962).

$$\frac{\partial(h\sigma_{11})}{\partial s_1} + \frac{\partial(h\sigma_{12})}{\partial s_2} + \rho g\,(e_1 . e_g) h - F_{fr} \cos\alpha_2 = 0$$

$$\frac{\partial(h\sigma_{12})}{\partial s_1} + \frac{\partial(h\sigma_{22})}{\partial s_2} + \rho g\,(e_1 . e_g) h - F_{fr} \cos\alpha_2 = 0$$

$$N - h\sigma_{11} k_1 - h\sigma_{22} k_2 + \rho g (e_n . e_g) h = 0$$

$$F_{fr} = c + fN \qquad (1)$$

Here $\mathbf{e}_g$, is a unit vector represented gravitational force direction; s_1 s_2- curvature coordinates of current point (Korn G.A., Korn T.M, 1968); $h=h(s_1, s_2)$ is a snow depth, measured perpendicularly to slope surface; σ_{ij} - stress tensor i,j =1,2; π - snow density; g - gravitational acceleration, $\cos\alpha_1$ and $\cos\alpha_2$ - directional cosines of displacement vector $\mathbf{u}=(u_1,u_2)$ in local basis $\mathbf{e}_1, \mathbf{e}_2$, $\cos\alpha_1 = \cos\alpha_2 = 0$ if and only if $|\mathbf{u}|=0$, F_{fr} - friction force between the snowpack and underlying surface, c-coefficient of cohesion, f - coefficient of friction.

System (1) should be completed by linear equations which couple the strains and stresses:

$$\sigma_{11} = \frac{E}{1-\nu^2} \left(\frac{\partial u_1}{\partial s_1} + \nu \frac{\partial u_2}{\partial s_2}\right)$$

$$\sigma_{12} = \frac{E}{1-\nu} \left(\frac{\partial u_1}{\partial s_2} + \frac{\partial u_2}{\partial s_1}.\right)$$

$$\sigma_{22} = \frac{E}{1-\nu^2} \left(\frac{\partial u_2}{\partial s_2} + \nu \frac{\partial u_1}{\partial s_1}\right) \qquad (2)$$

where E is α Young's modulus, ν is a Poisson ratio. According to (Nye J.K., 1959, Nefed'ev V.O and Bozhinsky A.N., 1989) this system of equation should be solved with Dirichlet boundary conditions $u_T=0$, where is a boundary of considered surface.

In order to solve this set of equations the knowledge of all snow parameters is required. Many field experiments demonstrate large spatial variability and uncertainty of such parameters as snow depth, coefficient of cohesion, coefficient of friction and snow density. This fact strongly motivates a stochastic description of the snowpack properties. Thus, other physical quantities in the model, the displacement vector $\mathbf{u}$ and stress tensor σ_{ij} also become stochastic. In the present study we assume that h, ρ, c are distributed as a Gaussian random field with a prescribed expectations, variances and covariation functions, that are found previously by a field measurements. Our aim is to find different statistical moments, for example, such useful statistical estimates as probability to exceed some threshold value of stress at every point of slope or probability density function of stress.

The stochastic solution of the problem in this study is obtained by the Monte Carlo simulation method. In this method, equations (1) - (2) are solved for a large number of realizations of h, ρ, c. From the large number of deterministic solutions for these realizations, any statistical moments are obtained.

PREPROCESSING

Because of extreme requirement to efficiency we choose a finite difference method to solve (1) - (2) numerically. In order to apply this method to our problem we should generate a mesh of curvature lines and build a random fields of parameters with prescribed distributions and covariations.

Let us assume that the slope surface has a form $z=f(x,y)$; x,y,z are Cartesian coordinates. The curvature lines may be found by solving an ordinary differential equations (Korn G.A., Korn T.M, 1968):

$$\begin{vmatrix} dy^2 & -dxdy & dx^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0 \qquad (3)$$

where $E=1+(\partial z/\partial x)^2$, $F=(\partial z/\partial x)\cdot(\partial z/\partial y)$, $G=1+(\partial z/\partial y)^2$,

$$L=\frac{\partial^2 z/\partial x^2}{\sqrt{1+(\partial z/\partial x)^2+(\partial z/\partial y)^2}} \qquad M=\frac{\partial^2 z/\partial x\partial y}{\sqrt{1+(\partial z/\partial x)^2+(\partial z/\partial y)^2}}$$

$$N=\frac{\partial^2 z/\partial y^2}{\sqrt{1+(\partial z/\partial x)^2+(\partial z/\partial y)^2}}$$

In practice the mountain slope represented as discrete set of samples (x,y,z). There is two approaches to calculate the coefficients of (3) - either using a B-spline interpolation or approximate a surface using quadratic or qubic surface. Both methods guaranty the existence of continuous first and second order derivatives.

In present study we used the B-splines package from Diffpack library (Xing Cai and Hans P. Langtangen, 1994). For information on it and on the conditions for its use one may send an E-mail to *dp-info@si.sintef.no*. The package offers tools for tensor product surfaces with emphasis on interpolation of discrete data and on features needed when using splines for solving differential and integral equation therefore simplifies the task of calculating the coefficients in (3). The programs and packages for surface fitting by using the polynomial of degree 2 or 3 are accessible via ftp from netlib. The pair of equations (3) are solved by adaptive Runge-Kutta method (William H. Press et al., 1992).

The fluctuating component of stochastic fields $\tilde{h}(s_1,s_2)$,$\tilde{c}(s_1,s_2)$ and $\tilde{\rho}(s_1,s_2)$ has correlation functions $\psi_h(s_1,s_2)$ $\psi_c(s_1,s_2)$ and $\psi_\rho(s_1,s_2)$ that are obtained by field measurements. Building a random field with given properties includes following steps.

- Generating of Gaussian δ - correlated spatial random fields $\tilde{h}_\delta(s_1,s_2)$,$\tilde{c}_\delta(s_1,s_2)$ and $\tilde{\rho}_\delta(s_1,s_2)$ with zero mean value and given variance;
- Transformation of derived δ - correlated spatial random fields from the space domain to the wavenumber domain using 2-dimentional fast Fourier transform (William H. Press et al., 1992);
- Transformation by the same way of $\psi_h(s_1,s_2)$,$\psi_c(s_1,s_2)$ and $\psi_\rho(s_1,s_2)$ to their power spectrum $\Psi_h(k_1,k_2)$ $\Psi_c(k_1,k_2)$ and $\Psi_\rho(k_1,k_2)$;
- Building the fluctuative component of random field in space domain by inverse Fourier transform of the functions , $h_\delta(k_1,k_2)\cdot\sqrt{\Psi_h(k_1,k_2)}$, $c_\delta(k_1,k_2)\cdot\sqrt{\Psi_h(k_1,k_2)}$ and so on;

After this operation we may obtain a required random field as a sum of mean value and fluctuation of any snowpack parameter at all nodes of mesh. As have been shown, the Cartesian coordinates of nodes may be computed using (3) Now we are ready to solve a problem (1) - (2) numerically.

Modelling of stress field and exemplifying results.

Substitution of finite difference representation of partial derivatives into (1) - (2) leads to large system of nonlinear equations with sparse matrix. We applied a simple iterative method to solve this system according to (Nefed'ev V.O. and Bozhinsky A.N., 1989) and calculate an exceedance rate of stress threshold value and stress distribution at arbitrary point of slope.

The sequence of field experiments in Khibiny yields the following empirical formulae for covariations of h,ρ,c:

$$\Psi_h(l)=0.38\cdot h_{mean}\exp(-0.009\cdot l^{1.5})$$
$$\Psi_\rho(l)=0.11\cdot\rho_{mean}\exp(-0.15\cdot l^{0.83})$$
$$\Psi_c(l)=0.28\cdot c_{mean}\exp(-o.16\cdot l^{1.02})$$

where . $l=\sqrt{s_1^2+s_2^2}$ We choose a typical values $h_{mean}=1\ m$, $\rho_{mean}=300 kg/m^3$, $c_{mean}=1000 N/m^2$ to demonstrate a results of modelling.

Other parameters assumed to be deterministic because a solution of problem is far less sensitive for their variations than for variations of h, ρ, c, so they are set to mean values, namely f=0.4, ν-0.3, $E=10^7$Pa.

Mountain slope geometry accepted for calculation of snow cover stability is shown in Fig. 1. This surface was obtained from real profile of mountain by fitting of quadratic polinomial. The rms residual does not exceed 20 m. Fig. 2 shows the probability field of exceeding of threshold value by σ_{thr}=12000Pa absolute value of σ, i.e.

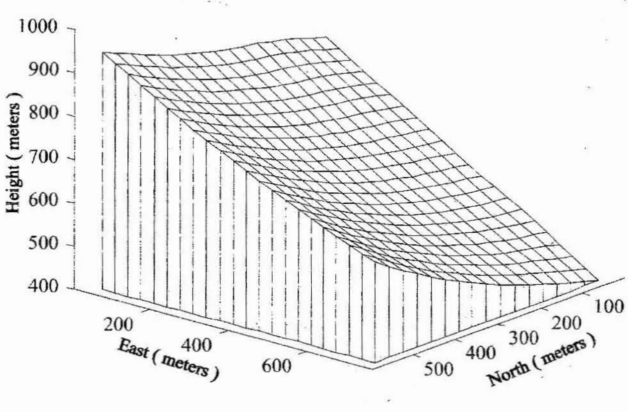

Fig. 1 Slope of the mountain

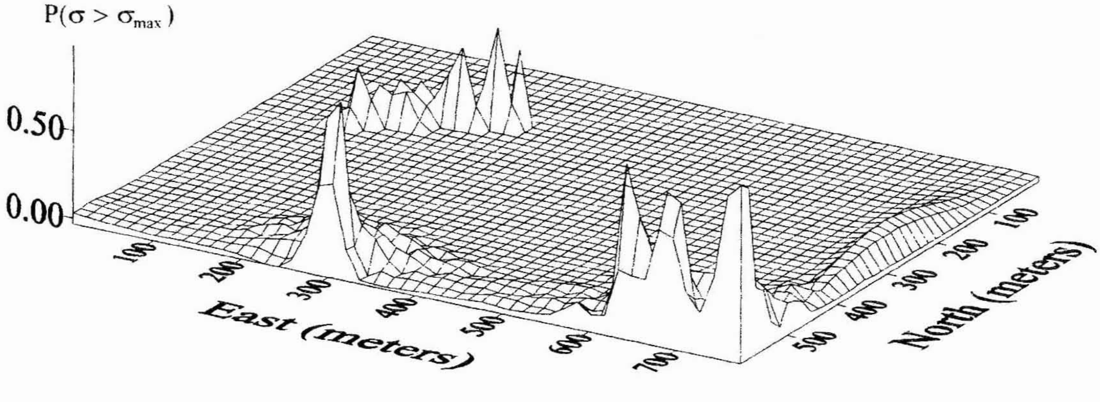

Fig. 2 Spatial probability distribution of exceeding a threshhold value of stress.

$P(|s|>s_{thr})$. It is clearly seen that the geometry of surface is a master factor that determines potentially dangerous zones.

Fig. 3 represents a distribution of stress at two arbitrarily choosen points - x=550, y=300 (top panel) and x=200,y=200. As far as the slope is steeper near first point, the distribution of stress here has a tail much longer than distribution of stress near second point.

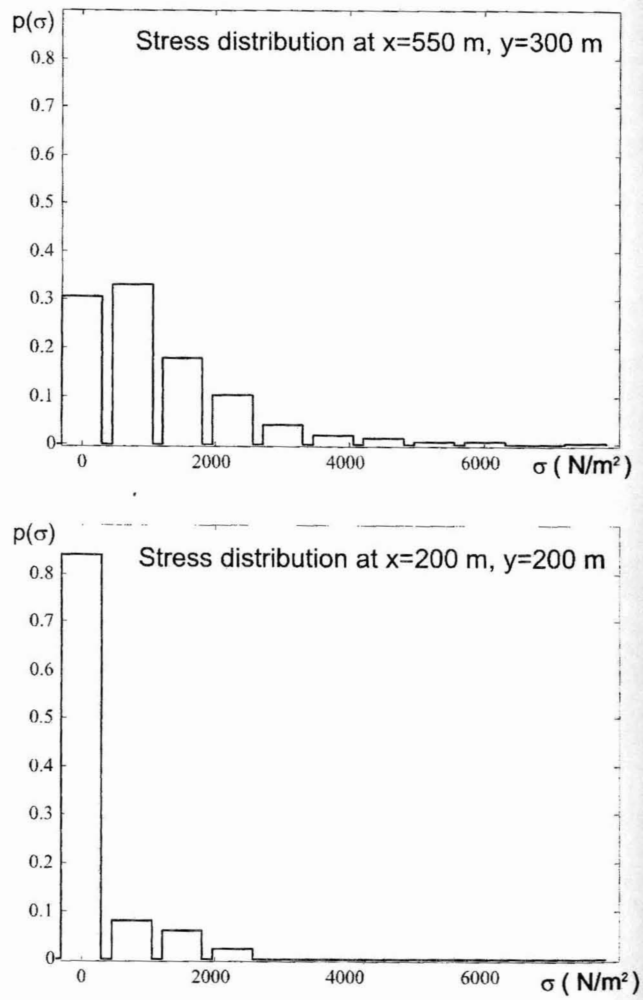

Fig. 3 Probability of stress at the point x=550 m, y=300 m (upper panel) and x=200 m, y=200m (lower panel).

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