Abstract.—Methods for modeling avalanche dynamics have evolved from the analogy with water flow in open channels. Voellmy (1955) pioneered the first general model for avalanche speed and runout distance. Subsequent researchers have extended the one dimensional, steady state theory to transient flow and quasi-steady two dimensions. Recently, numerical methods have been developed to simulate time-dependent two-dimensional, laminar dense flow and turbulent powder avalanches.

This paper presents a numerical model of three dimensional turbulent avalanche dynamics. The avalanche is treated as a Newtonian, inhomogeneous, multiphase gravity current in which density stratification is determined by the suspended snow particle distribution. Primitive equations for mass snow volume fraction, and momentum are written with density variation terms retained (i.e., no Boussinesq approximation). A two-equation, second order closure k-e turbulence model for density affected flows is used to complete the Reynolds averaged bulk flow equations. Results of the model initialization scheme for a typical powder avalanche are presented and numerical methods for solving the field equations are outlined.

1.0 INTRODUCTION

Hundreds of thousands of avalanches fall in the mountains of the world each winter. Most pose no threat to man or his structures. However, the expansion of winter recreational facilities, housing subdivisions, highways, mining operations, and energy lifelines in mountainous areas has increased the frequency of exposure to avalanches. While the number of avalanches that occur during a given winter does not change significantly, the growing exposure makes avalanches increasingly hazardous (Tesche and Yocke, 1976; Armstrong and Williams, 1986).

Fortunately, there are ways to mitigate avalanche hazards (de Quervain, 1966; Mellor, 1968; Perla and Martinelli, 1976; LaChapelle, 1977; Tesche, 1977). Passive control measures such as flow diversions, snowsheds, snow fences, closures, and zoning ordinances, and active control measures such as blasting and snow packing have been used in the United States and in Europe for decades. Of the various methods for mitigating avalanche disaster, restricting access to hazardous areas during periods of high danger, prohibiting construction in avalanche-prone areas, and designing structures that can withstand avalanche impact are the most reliable.

Structural design sufficient to survive avalanche impact requires a means of estimating avalanche mass, velocity, and forces (static and dynamic). The goal of the research summarized in this paper is to develop a three dimensional, time-dependent model for turbulent powder snow avalanches that may be used as a tool for exploring the various facets of the phenomena through numerical experimentation. Ultimately, such a model may prove useful in avalanche zoning and structural design situations.

This paper is organized as follows. In Section 2.0, analytical and numerical models developed over the last thirty years for describing avalanche motion are summarized. Section 3.0 presents a conceptual model of the turbulent powder avalanche. Emphasis is placed on the basic physical processes important in the development of a powder avalanche (e.g., air and snow entrainment, suspension, turbulence, density stratification, sedimentation, and dissipation). Formulation of the conceptual model is guided by laboratory and field experiments involving avalanches, density currents, and turbidity.


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currents. Based upon the conceptual model, in Section 4.0 the governing dynamic equations for the three-dimensional transient powder avalanche are derived. Section 5.0 introduces the turbulence closure scheme developed recently by Rodi (1980, 1985) for density stratified flow. Boundary conditions (i.e., entrainment at the surface, front, and a loft and transitional flow near the bed) are addressed in Section 6.0. In Section 7.0, numerical solution issues such as linearization of the turbulent Navier-Stokes equations, discretization, grid mesh generation, and numerical integration are presented. A brief summary is offered in Section 8.

2.0 REVIEW OF AVALANCHE DYNAMICS MODELS

This section summarizes research over the last thirty years in developing models for quantifying avalanche velocities, pressures, and related quantities.

Avalanche Motion

Despite the many names given to different forms of snow avalanches (Seligman, 1936), two principal types of motion are commonly recognized. One is the surface (dense-snow or flowing) avalanche and the other is the airborne powder avalanche. (Classification on the basis of release mechanism yields loose snow and slab avalanches.)

The surface avalanche is essentially a large bulk-density, granular flow that moves along the terrain surface. The airborne powder avalanche, by contrast, is a highly turbulent two-phase, density-driven flow. Typical ranges of bulk densities for flowing and powder avalanches are 0.1-0.4 gm/cm$^3$ and $\leq$0.10 gm/cm$^3$, respectively. One of the characteristics distinguishing flowing versus powder avalanches is the amount of air entrained. Early avalanche modelers focused upon flowing avalanches due to their close analogy to homogeneous, hydraulic flows. Only recently have the principles of multiphase granular flow been coupled with turbulence modeling techniques to attack the problem of simulating powder avalanches rigorously.

Salient features of important modeling studies are presented next. Models reviewed here are those developed by Voellmy, Salm, Shen and Roper, Soviet researchers, Hopfinger and Tochon-Danguy, Beghin and Brugnot, Brugnot and Pochat, Perla et al., Lang et al., and Scheiwiller et al. Other reviews of avalanche models are discussed by Perla (1980), Hopfinger (1983), and Scheiwiller and Hutter (1982).

Voellmy’s Avalanche Model

Voellmy (1955) is widely credited as the first to systematically apply the principles of fluid mechanics to the study of avalanche dynamics (LaChapelle, 1977; Perla, 1980). He conceptualizes avalanche motion as being analogous to steady-state open channel hydraulic flow. A one dimensional, uniform flow momentum balance is written as:

$$\frac{d\tau}{dz} + \frac{g(\rho - \rho_a)\sin \theta}{a} = 0$$

(1)

where $\rho$ is the avalanche (snow and air mixture) density, $\rho_a$ is the air density, $\theta$ is the slope angle, and $\tau$ is the fluid shear stress at the bed. (Hereafter, we refer to the interface between the flowing avalanche and the upper stationary snow or ground surface as the bed.) Assuming the lower boundary condition at the bed consists of a dynamic drag and a frictional force and the upper boundary condition at $z=H$ consists only of dynamic drag, Voellmy integrates Equation (1) to obtain his classic expression for maximum avalanche flow velocity:

$$V_{\text{max}} = \left[\frac{\xi h(\sin \theta - \mu \cos \theta)}{\xi h_0} \right]^{1/2}$$

(2)

where $\xi$ is the sum of density and drag coefficients, and $\mu$ is the coefficient of dynamic friction. The friction term involving $\mu$ is added to the conventional hydraulic theory to account for avalanche acceleration on steep slopes ($\tan \theta > \mu$) and deceleration on gentle slopes ($\tan \theta < \mu$).

From a simple energy balance, Voellmy then develops a formula for the runout distance, i.e. the downslope distance from the location of maximum velocity to the point of avalanche deposition. The result is:

$$S = V_{\text{max}}^2 \left[ \frac{2g(\mu \cos \theta - \tan \theta)}{V_{\text{max}}^2 + 2g/\xi h_0} \right]$$

(3)

$S$ is measured from the reference position on the slope where the maximum velocity is given by Equation (2). Voellmy’s concept of an avalanche reflects the endless-fluid viewpoint (Perla, et al., 1980) in that a steady state motion is maintained by a continuous supply of snow forming the trailing body of the avalanche.

Voellmy’s expressions and refinements to them form the basis for many of the engineering calculations of maximum avalanche velocities performed in the last thirty years (see, for example, Leaf and Martinelli, 1977; Perla and Martinelli, 1976; Mellor, 1978; Shearer, 1973, 1975; Buser and Frutiger, 1980). Much of the effort in using and extending Voellmy’s pioneering model has been in selecting appropriate friction and drag coefficients, $\mu$ and $\xi$. Numerous investigators have suggested a range of values for $\xi$ (Perla, 1980). For example, in their engineering manual, Leaf and Martinelli (1977) give a range of $\xi$ values, from 500 m/sec for very rough slopes to 1,800 m/sec for smooth slopes. There is also a wide range in estimated values of $\mu$ in the literature, based both on field observations (Heimgartner, 1977; Schaefer, 1975) and modeling studies (Lang et al., 1979a; Martinelli, Lang, and Hears, 1980).

Apart from the practical difficulties of establishing unambiguous values for $\mu$ and $\xi$, Voellmy’s method is confounded by the need to
established the reference position that separates acceleration and deceleration portions of the avalanche track. Perla et al. (1980) propose an alternative two parameter model based upon the same governing equations. Their model (presented later) avoids this internal inconsistency in Voellmy's model. Despite its limitations, however, many engineering studies continue to use Voellmy's model because of its simplicity and the considerable experience developed over three decades in fitting the model equations with empirical data. A skilled practitioner is often needed to properly choose the appropriate friction parameters based upon prevailing snow and terrain conditions (Dent, 1986).

Salm's Improvements

Salm's (1966, 1968) theoretical studies significantly add to Voellmy's pioneering work. Conservation of momentum across an arbitrary section $F$ is written as:

$$
\frac{\partial \gamma \frac{dy}{dt}}{\partial x} = \left[ \gamma F \sin \theta - \mu \gamma (F \cos \theta + S) \right] \frac{\gamma - \gamma_L}{\gamma - \gamma_L}
$$

$$
- \frac{\alpha F}{s} \left( \frac{n U}{\delta} V - \frac{U V}{2} \right) - \frac{C \gamma F \gamma_L}{2 g s} V^2
$$

where:
- $\gamma_L$ = specific gravity of air
- $\gamma$ = mean specific gravity of the avalanche
- $\mu$ = coefficient of friction
- $S$ = pressure on the lateral sides of the avalanche
- $s$ = length of the avalanche
- $\omega$ = fraction of frontal snowpack area $F$ disturbed by approaching the avalanche
- $\beta$ = breaking stress of the snow parallel to the snow surface
- $\eta$ = viscosity of snow
- $U$ = perimeter of the sliding surface
- $k$ = roughness factor
- $C_w$ = avalanche drag coefficient
- $g$ = acceleration of gravity
- $\delta$ = boundary layer thickness

Salm obtains a closed form solution upon integrating Equation (4) but also develops a much simpler formula for the velocity of a flowing avalanche as:

$$
v = \left( a R k^2 \right)^{1/2} \tan \left[ \frac{\gamma - \gamma_L}{\gamma - \gamma_L} \left( \frac{2 a R k^2}{R k^2} \right)^{1/2} \right]
$$

where $R$ is the hydraulic radius, and $a$ is a function of slope geometry and friction coefficients. Among Salm's main contributions is the postulation of three components of friction in an avalanche: a force independent of velocity, a force proportional to velocity, and a force proportional to the square of the velocity. These forces are attributed to static friction, viscosity, and turbulence, respectively. He also proposes that surface waves do not propagate in a flowing avalanche.

Shen and Roper

Shen and Roper (1970) develop a two-dimensional model of the powder avalanche based upon fluid mechanics principles and find that Voellmy's velocity equation (Equation 2) agrees with experimental data derived from density current studies. They also find that Voellmy's suggested range for the turbulent friction coefficient (400 m/s² to 600 m/s²) gives a reasonable estimate of terminal velocities for avalanche flow over smooth bed surfaces.

Shen and Roper (1970) model the mature powder snow avalanche as a two-dimensional, steady state density current, expressing the continuity and momentum equations as:

$$
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
$$

$$
\rho \left( \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = (\rho - \rho_a) g \sin \alpha + F_t + F_b
$$

where $F_t$ and $F_b$ are friction forces acting on the top and bottom surfaces of the avalanche, respectively. Depth-integrating Equation (7) gives

$$
g \sin \alpha (\rho - \rho_a) H = \left( f_a + f_b \right) \rho \frac{u^2}{8}
$$

where $H$ is the height of the avalanche front and $f_a$ and $f_b$ are the air-snow and bed surface friction factors, taken to be 0.085 and 0.015, respectively.

Rearranging Equation (8) gives an expression for maximum (terminal) avalanche velocity very similar to Voellmy's:

$$
u = \frac{2g \sin \alpha \left[ \rho - \rho_a \right]}{(f_a + f_b) \rho} H
$$

Shen and Roper (1970) suggest that the avalanche flow height, $H$, can be estimated adequately by Voellmy's procedure:

$$
H = \frac{2 \rho' \rho_a}{\rho_a \sin \alpha} (h + h')
$$

where $\rho'$ = avalanche density, $h$ = depth of the flowing snow layer, and $h_a$ = depth of the entrained snow layer. They give typical values of $\rho'$ and $\rho$ of 200 kg/m³ and 1.25 kg/m³, respectively.

Soviet Researchers

Avalanche modeling in the Soviet Union has also explored numerical methods, perhaps best exemplified by the work of Kulikovskii and Egit (1973), Gregoryen and Ostrovmov (1977), Brugnot and Pochat (1981), and Egit (1983). Assuming an analogy with one dimensional unsteady open channel hydraulics, this group of researchers typically write the continuity and momentum equations as:
Hopfinger and Tochon-Danguy

Hopfinger and Tochon-Danguy (1977) and Tochon-Danguy and Hopfinger (1975) discuss the similarity requirements for modeling avalanche flow in the laboratory. They present arguments for treating the powder snow avalanche as a Newtonian fluid for which the Boussinesq approximation is inappropriate, citing typical density ratios of order 10 or more. Beginning with two-dimensional, steady flow equations for a gravity current:

\[ \frac{\partial \rho u}{\partial t} + \frac{\partial (\rho u^2)}{\partial x} + g \rho \frac{\partial h}{\partial x} \cos \theta = 0 \]  

(11)

\[ \frac{\partial (\rho Av)}{\partial t} + \frac{\partial (\rho Av v)}{\partial x} + g \rho \frac{\partial h}{\partial x} \cos \theta = \frac{\partial}{\partial x} \left[ \sin \theta - \mu \cos \theta - \frac{v^2}{R} \right] \]  

(12)

where \( R \) is the hydraulic radius and \( A \) is the cross-sectional area. Hydrostatic pressure is assumed. The equation set is completed with the addition of an equation for hydraulic jump conditions at the front of the avalanche. The Soviet models require the knowledge of (1) surface and internal friction coefficients, (2) conditions at the avalanche head, and (3) the influence of velocity and pressure fields on the snow-air density distribution.

Beghin and Brugnot (1983) consider the powder snow avalanche to be a finite-volume buoyant thermal flowing down an incline. They derive an expression for the front velocity, \( U_f \), of the avalanche as:

\[ U_f = C \left[ \frac{gsin\theta}{\rho E} \left( C_2 \Delta \rho N N + \Delta \rho A o / x \right) \right]^{1/2} \]  

(16)

where:
- \( C \) = constant
- \( E \) = snow entrainment coefficient
- \( C_2 \) = empirical coefficient
- \( \Delta \rho N \) = density excess of the undisturbed snow ahead of the avalanche
- \( h_N \) = height of the snow ahead of the avalanche
- \( A_o \) = initial volume of the avalanche
- \( \Delta \rho o \) = initial density excess of the avalanche, and
- \( \alpha \) = slope angle.

This expression for \( U_f \) is valid only when the snow concentrations within the avalanche are low (i.e., for a Boussinesq fluid).

Beghin and Brugnot (1983) consider the powder snow avalanche to be a finite-volume buoyant thermal flowing down an incline. They develop formulae for the velocity and density based on simplified mass and momentum equations and then compare model predictions with laboratory observations made in an inclined water tank using a salt solution and sand and barium sulfate. The momentum equation is written as:

\[ \frac{dU_f}{dx} = - \Delta \rho A \sin \theta - U_f^2 \frac{d}{dx} \left[ (K \rho_a + \rho) A \right] \]  

(17)

where the first term is the momentum change over time interval \( dt \), the second term represents the slope-parallel component of the gravity force, and the last term of Equation (17) is a resistive force due to snow and air entrainment. Beghin and Brugnot assume that the height and length of the avalanche grow at rates \( a_1 \) and \( a_2 \) that are independent of slope distance and can be estimated empirically. Assuming the avalanche longitudinal profile resembles an elliptic half-cylinder, they obtain the following expressions for frontal velocity and avalanche density:

\[ U_f^2 = 2g \sin \theta \int_{x_1}^{x_f} \left( \beta_1, \beta_2, \beta_3, \beta_4 \right) i=0,4 \]  

(18)

where \( \beta_1, \beta_2, \beta_3, \beta_4 \) are initial and dynamic growth coefficients for the avalanche, and:

\[ \rho - \rho_a = \frac{C_1 h \rho_o x_f + (\rho_o - \rho_a) k L}{h_0 + a_4 x_f} \]  

(19)

where \( L_0 \) and \( H_0 \) are the initial length and height of the avalanche, respectively. At large downslope distances, \( U_f \) has an asymptotic value proportional to \( (C_1 h \rho_o)^{1/2} \) assuming no snow is entrained.

Brugnot and Pochat (1981) formulate a quasi-one dimensional avalanche model and then discuss a number of sensitivity tests examining the effect on front velocity due to variations in input parameters (e.g., static, laminar, and turbulent friction coefficients, track geometry, snow entrainment rate). They consider an avalanche flowing down a gully whose wetted cross-sectional area is:

\[ s = kh^n \]  

(20)

where \( h \) is the avalanche flow height and \( k \) and \( n \) are variables related to the gully geometry. Defining the snow mass flux \( P = VS \) where \( S = \rho s \), they derive the following equations:

\[ \frac{\partial S}{\partial t} + \frac{\partial P}{\partial x} = 0 \]  

(21)
At the front of the avalanche, they assume a hydraulic jump, for which the following equations hold:

\[
\frac{\partial P}{\partial t} - \frac{\partial S}{\partial t} \left[ gh - \frac{P^2}{S^2} \right] \frac{\partial S}{\partial x} = \text{constant}
\]

\[
- \left[ f_s g S \cos \theta + f_p \frac{P}{R} + f_t \frac{P^2}{SR} \right]
\]

(22)

This equation can be rewritten as:

\[
Mv \frac{dv}{dx} = M[g \sin \theta - \mu g \cos \theta] - \frac{1}{2} C_D \rho_a \left( \rho g \left( \frac{\partial P}{\partial x} \right) + \frac{dM}{dx} \right)^2
\]

(26)

The term \( \frac{dM}{dx} \) is the snow entrainment along the track and acts to retard avalanche acceleration. Combining all velocity-squared terms gives the so-called drag parameter, \( D \), where:

\[
D(s) = \frac{\mu M}{r} + \frac{dM}{dx} + k
\]

(27)

Here \( r \) is the local radius of curvature.

Substitution into Equation (26), yields the following differential equation describing the motion of the avalanche mass center from release to runout along a slope of arbitrary geometry:

\[
\frac{1}{2} \frac{dv^2}{dx} = g (\sin \theta - \mu \cos \theta) - \frac{D}{M} v^2
\]

(28)

Although fairly simple, the centre-of-mass model describes the major features of avalanche acceleration/deceleration provided key parameters can be computed or specified. These include the drag coefficient, \( C_D \), density stratification \( \rho / \rho_0 \), mass \( m \), friction \( \mu \), and entrainment rate, \( dM/dx \). The drag coefficient and surface entrainment rate are probably the two most difficult parameters to determine.

Perla et al., (1984) extend the centre-of-mass model concept to cover an ensemble of discrete snow particles. Abandoning the continuum approach, they include a stochastic term, \( R V \), in the governing force-momentum equation (Equation 26) and calculate the sign of the random term through Monte Carlo simulation. Thus a third adjustable parameter, \( R \), is added to the two parameter model. Stochastic modeling enables treatment of snow entrainment at the avalanche front and variation in resistance parameters as a function of position and velocity.

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Perla and Co-workers

In a series of eight closely-related papers, Lang and co-workers present the theoretical formulation and practical applications of a numerical model of snow avalanche motion and impact (Lang et al., 1979a, b; Lang and Martinelli, 1979; Pederson et al., 1979; Lang and Brown, 1980; Martinelli et al., 1980; Dent and Lang, 1980; LaChapelle and Lang, 1980; Lang and Dent, 1980). The AVALANCHE code is specifically designed to treat the flow of the denser core of a flowing avalanche; the airborne component is not modeled.
The premise underlying the AVALANCHE model is that many avalanches exhibit characteristics of rotational laminar flow, with turbulent or rigid-block cases being exceptional extremes (Lang et al., 1979b). Based upon this laminar flow conceptualization, the two dimensional, time-dependent Navier-Stokes equations are written as:

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= \frac{\partial p}{\partial x} + \nabla^2 u \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} &= \frac{\partial p}{\partial y} + \nabla^2 v
\end{align*}
\]

For mass continuity (assuming incompressibility) they write:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

The model equations are solved using a highly generalized numerical fluid dynamics code originally developed at Los Alamos National Laboratory (Birt et al., 1975) for a broad range of laminar, free surface flows.

Two physical parameters are required by the AVALANCHE model—the surface friction factor, \( f \), and the kinematic viscosity, \( \nu \). The surface friction factor enters in the lower boundary condition and is also used to control the rate at which the avalanche decelerates in the runout zone. The fluid viscosity, \( \nu \), is treated as a constant over the whole path.

Several features of avalanche motion are potentially modeled by AVALANCHE including snow entrainment, transient behavior, arbitrary track geometries, and a variable free surface. Since most avalanches are turbulent, however, (de Qvervain, 1966; Grigoryan, 1975; Hopfinger, 1983; LaChapelle, 1977; Mellor, 1968; Mears, 1976; Perla, 1980; Perla and Martinell, 1976; Seligman, 1936; Scheiwiller and Hutter, 1982), the AVALANCHE model is restricted in its range of applicability to low Reynolds number, surface avalanches.

Notwithstanding this theoretical limitation, the AVALANCHE code has been able to reproduce observed runout distances for avalanches over a wide variety of snow and terrain conditions. In applications by an experienced user, and given appropriate selection of model inputs for friction, viscosity, and initial slab thickness, good agreement between runout distance predictions and observations can be obtained for surface avalanches. Like Voellmy's and Perla's models, AVALANCHE does not supply unique solutions for velocity and runout distance.

Scheiwiller and Co-workers

Recently, Scheiwiller and Hutter (1982, 1983) and Scheiwiller (1986) published a detailed mathematical model formulation for powder snow avalanches based upon the dynamical theory of multiphase flow (Soo, 1967; Savage, 1979). This work constitutes an important advancement in avalanche dynamics modeling. An avalanche is viewed as a two-phase mixture (air and snow) as shown in Figure 2. Mass and momentum conservation equations are written for each phase and constitutive relationships for interphase friction are proposed. Closure is achieved with a frequently-used k-\( \epsilon \) turbulence model (Lauder and Spaulding, 1974). The governing equations are solved for steady, two dimensional flow on an inclined planar surface. As shown later in Sections 4 and 5, there are similarities between the conceptual approach taken by Scheiwiller and coworkers and that followed in the present research.

![Figure 2a. Conceptual framework for Scheiwiller's two-phase avalanche dynamics model--depiction of gravity, friction, entrainment, and turbulence phenomena. (Source: Scheiwiller, 1986).](image)

![Figure 2b. Conceptual framework for Scheiwiller's two-phase avalanche dynamics model--postulated velocity and density profiles.](image)
Scheiwiller (1986) derives mass and momentum conservation equations for air and snow phases assuming continua for both. For a given phase, \( v \), the mass and momentum balances are written as:

\[
\frac{\partial}{\partial t} \rho_v + \frac{\partial}{\partial x_i} (\rho_v u_v) = 0 \tag{32}
\]

\[
\frac{\partial}{\partial t} (\rho_v u_v) + \frac{\partial}{\partial x_i} (\rho_v u_v u_v) = \frac{\partial}{\partial x_i} (-f_x p) + \frac{\partial}{\partial x_i} (-\rho_v u_v u_v) + \rho g_v - m_v \tag{33}
\]

where:
- \( p \) = pressure
- \( m \) = momentum transferred to phase \( v \)
- \( f^v \) = volume fraction of phase \( v \)
- \( \rho_v \) = density of phase \( v \)
- \( - (\rho_v u_v u_v) \) = Reynolds stress tensor

Assuming no net momentum production:

\[
\sum_v m_v = 0 \tag{34}
\]

\[
m_2 = -m_1 \tag{35}
\]

Also, assuming that the density of phase \( v \) may be written in terms of the phase volume fraction, \( f_v \), and the phase material density, \( \langle \rho \rangle^v \), (denoted here by brackets):

\[
\rho_v = f_v \langle \rho \rangle^v \tag{36}
\]

and that the sum of the volume fraction is unity, viz:

\[
\sum_v f_v = 1 \tag{37}
\]

then the assumption of incompressible flow for both phases yields \( \langle \rho \rangle = \text{constant} \). Note that the phase densities for air and snow may vary due to variations in their respective volume fractions, but the material densities of air and snow are constant.

Scheiwiller (1986) poses a simple constitutive relationship for the interphase friction term, \( m_v \), as follows:

\[
-m_2 = + m_1 = C_{\text{frict}} \frac{1}{\tau} \left( <u_i^1> <u_i^2> \right) \tag{38}
\]

where \( \tau \) is a time scale.

Applying the above assumptions to steady, two-dimensional flow down an inclined chute gives:

\[
\frac{\partial f_1}{\partial t} + \frac{\partial}{\partial x_1} (f_1 u_{1j}) = 0 \tag{39}
\]

\[
\frac{\partial}{\partial t} (f_1 u_{1j}) + \frac{\partial}{\partial x_1} (f_1 u_{1j} u_{1j}) = -\frac{1}{\langle \rho \rangle^2} \frac{\partial}{\partial x_1} (f_2 p) + C_{\text{frict}} \left( \frac{1}{\langle \rho \rangle^2} \right) \left( \frac{1}{\tau} \right) \left( <u_i^1> - <u_i^2> \right) \tag{40}
\]

\[
+ f_1 \frac{\rho}{\langle \rho \rangle} \frac{\partial}{\partial x_3} \left( \frac{\partial u_{1i}}{\partial x_3} \right) \tag{41}
\]

\[
\frac{\partial}{\partial t} (f_2 u_{2j}) + \frac{\partial}{\partial x_1} (f_2 u_{2j} u_{2j}) = -\frac{1}{\langle \rho \rangle^2} \frac{\partial}{\partial x_1} (f_2 p) - C_{\text{frict}} \left( \frac{1}{\langle \rho \rangle^2} \right) \left( \frac{1}{\tau} \right) \left( <u_i^1> - <u_i^2> \right) \tag{42}
\]

where \( \mu_{t,1} \) and \( \mu_{t,2} \) are turbulent eddy viscosities that are determined from the turbulence model developed by Launder and Spalding (1974) and Spalding (1982).

For boundary conditions, Scheiwiller ignores snow entrainment at the ground surface and assumes a no-slip condition. Thus the mass of snow in the avalanche remains constant for the duration of the event. At the top boundary, an empirical air entrainment rate equation is prescribed.

Integration of the governing equations was originally attempted with two methods: (1) a finite difference algorithm, and (2) the Kantorowich technique. This latter method avoids dynamic modeling of vertical mass and momentum transfers by assuming shape profiles which can be used to hueristically distribute vertically such quantities as density, particle phase volume fraction, and velocity. Excessive numerical diffusion in the original finite difference scheme made it difficult for Scheiwiller to reproduce the observed steep vertical gradients in particle phase volume fraction so the finite difference method was dropped and the Kantorowich method was adopted for the model solution technique. This renders the entire modeling approach quasi-two dimensional since the dynamic flow equations are solved explicitly only in the downslope (i.e., \( x \)-coordinate direction).

Scheiwiller (1986) also presents the mathematical theory for quasi-two dimensional flow of the surface avalanche regime. Mass and momentum equations similar to Equations (39) through (42) are written, and an advective flux entrainment function is specified at the bottom surface to
account for air fluidization as the avalanche matures from surface motion to an airborne powder avalanche. At this writing, coding of the surface avalanche model is underway.

3.0 CONCEPTUAL MODEL OF POWDER AVALANCHE

A number of good reviews have discussed the various physical processes believed to be important in the initiation and maintenance of a powder avalanche (Mellor, 1968, 1978; Mears, 1975, 1976; Perla, 1980; Scheiwiller, 1986). In this section, several characteristics of the powder avalanche that influence the mathematical modeling of the phenomena are introduced. The physical processes are discussed within the temporal framework of the avalanche event (i.e., beginning with release and ending with runout and sedimentation). The discussion begins with the initial motion of an individual snow pack slab immediately after release, leaving to others the difficult task of explaining the causes and mechanisms of snow pack failure.

Avalanche Release and Initial Motion

Immediately upon failure, the snow mass moves principally as a unit sliding upon a few individual shear planes. Early Russian attempts to describe the dynamics of avalanches assumed that the snow continues to move as a rigid body (Moskalev, 1965, 1966). Other Russian investigators, however, have criticized this approach (e.g., Losev, 1965; Grigorian, et al., 1967, Briuchanov, et al., 1967) maintaining, as do most western researchers, that avalanche motion is best considered within the framework of fluid mechanics. Nonetheless, some elements of the kinematic viewpoint are used today (i.e., the centre-of-mass model). Once the snow mass is in motion, no-slip conditions on the bed surface generate a strong average shear and a normal stress which rapidly ruptures intergranular bonds. The disruptive effects of the non-uniform terrain surface disintegrates bonding between snow slab strata, forming irregularly shaped clods of snow which flow along innumerable shear planes. Waves and surges in the flowing snow cover are observable in this initial stage which often resembles a debris flow. Entrainment of ambient air is negligible. In this initial stage, the surface avalanche is analogous in many respects to underwater turbidity currents in which soil mass wasting is immediately followed by a subaqueous debris flow (Hampton, 1972).

Two steps are critical in the development of a powder avalanche from a flowing (or surface) avalanche—fragmentation of the initial sliding mass and air entrainment. Clearly, not all avalanches evolve into the powder avalanche; large liquid water contents, short fall distances, and other factors can constrain the avalanche to remain in the dense snow, flowing stage. However, if pulverization of the initial snow mass occurs and if a significant mechanism of air entrainment is operative, then powder snow avalanche development occurs. From experimental studies (Shoda, 1966, for example), it is known that the powder avalanche almost always occurs with an associated surface flow avalanche. Dry flowing avalanches (with densities somewhere between 50 kg/m³ to 300 kg/m³) can occur from initial snowpack conditions that can also produce low density (i.e., 15 kg/m³ or less) powder avalanches.

Phases of Motion

Three phases (or regimes) of motion in the powder avalanche can be identified, corresponding to different portions of the avalanche path (see Figure 3). Within and downslope of the starting zone, the avalanche is in an acceleration regime. When the sliding snow mass enters the track, a quasi-steady state regime evolves in which the processes of snow entrainment and sedimentation compete. As the terrain gradient diminishes, the avalanche emerges into the runout (or deposition) zone where expulsion of entrained air, snow particle sedimentation, and energy dissipation result in deceleration and ultimate cessation of motion.

Figure 3. Definition sketch for an avalanche path. (Source: Bovis and Mears, 1976)

The first two phases of motion of the avalanche are thought to be analogous to those observed in gravity current experiments in the laboratory. Rottman and Simpson (1983, 1984), for example, describe a preliminary adjustment phase during which initial conditions are important and a subsequent self-similar phase in which the gravity current front speed decreases as $t^{-1/3}$. 

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The principal mechanism(s) responsible for transition from a dense snow flowing avalanche to a powder avalanche remain uncertain. For turbidity currents, Morgenstern (1967) and Shreve (1968) attribute the transition to an increase in turbulence levels due to fluidization, while Van Andel and Komar (1969) speculate that a hydraulic jump separates the high density super critical sliding phase from the low density, subcritical turbidity current phase. Laboratory simulations of density current flow suggest that air entrainment is the principal mechanism for transition from a surface to an airborne-type flow regime. Since existence of hydraulic jumps in avalanches have not been unambiguously documented in the avalanche literature, it seems that the air fluidization theory is the more plausible.

The flowing avalanche becomes fluidized when the upward velocity component of the turbulent eddies exceeds the snow particle free fall velocity in the flow. This criterion for snow suspension is used in blowing snow studies (Mellor, 1965; Pomeroy and Male, 1985; Schmidt, 1982), and in describing sand particles in a turbulent air stream (Bagnold, 1962, 1966). Furthermore, very recent studies by Scott and Carter (1986) reveal similar momentum effects in blowing snow and sand. Since the characteristic turbulence velocity is an order of magnitude smaller than the mean avalanche flow speed, for a typical powder avalanche this implies that snow particles of several millimeters effective diameter are easily suspended in the turbulent flow. Mears (1976) studied a large number of dry-slab avalanche deposition zones and concluded that most of the mass within this type of flowing avalanche was in size fraction too large to be suspended by turbulence. Accordingly, the powder avalanche regime was not obtained in the slab cases he examined.

Entrainment of only a small amount of air may result in a significant reduction in the cohesiveness of the original slide mass with a consequent increase in the fluid-like behavior of the flow. Incorporation of air directly increases the fluid volume, producing larger average particle spacings and significantly reduced interphase friction forces.

There are strong similarities between the flow of non-cohesive granular mixtures (see, for example, Woodcock and Mason, 1978; Savage, 1979; Shen 1985a,b; Nuziato, 1983) and dry avalanches (Hutter and Scheiwiller, 1983; Dent, 1986). Furthermore, avalanches exhibit characteristics closer to turbidity currents compared with the related gravity current phenomena. Turbidity currents, like avalanches, possess an added degree of freedom in that the solid constituent responsible for the density difference is not constant but is entrained and deposited throughout the course of the flow. In this sense, the general class of turbidity currents (including avalanches) can be said to be buoyancy-generating or self-accelerating (Garcia et al., 1986).

Once the avalanche enters the track and sufficient air entrainment has occurred to produce a dilute, turbulent powder avalanche cloud, a quasi-equilibrium condition is attained in which the rate of new snow entrained into the avalanche through various mechanisms roughly balances the sedimentation loss of material to the bed surface. Since the relative contribution of snow entrainment and deposition losses to the overall mass budget is probably quite site- and condition-specific, most models assume either an exact cancellation of mass gain and loss or a linear (with distance) mass entrainment rate.

As the runout zone is approached, the turbulence intensity is reduced and the granular snow material settles through the pore fluid (air), compacts, and stops. Stated differently, expulsion of pore fluid does not occur in a powder avalanche until the fluid phase loses its capacity to support the granular solids.

**Boussinesq Approximation**

For an incompressible medium, Boussinesq (1903) demonstrated that density fluctuations associated with admixtures of fluid of different specific weights do not appreciably affect the momentum balance except with respect to terms involving the gravitational acceleration. The Boussinesq approximation leads to important simplifications of the equations of motion and associated numerical solution procedures; it is widely invoked in fluid dynamics. Spiegel and Veronis (1960) showed that for a shallow layer of fluid, the Boussinesq approximation can be derived without the restriction of incompressibility. Calder (1968) restated and elucidated this derivation. However, the conditions of validity of the Boussinesq approximation are more complicated than they initially appear. Mahrt (1986) presents a rigorous analysis of the Boussinesq approximation and its implications, although his analysis is focussed on atmospheric flows.

Few researchers discuss explicitly the conditions for which the Boussinesq approximation holds in an avalanche. Hopfinger and Tochon-Danguy (1977) state that the approximation does not hold for an avalanche, citing density ratios of order 10 or more, compared with their laboratory flows with salt or sand suspensions where the density ratio was 1.01 to 1.15. [The density ratio is defined here as the ratio of the avalanche (in nature or in the laboratory) average density to that of the ambient fluid. This ratio is given several different definitions in the literature.] Scheiwiller (1986) develops non-Boussinesq equations for his quasi-two dimensional powder avalanche model but does not discuss the implications imposed by this added complexity. Tochon-Danguy and Hopfinger (1975) also indicate that powder avalanches are non-Boussinesq due to large density ratios, unlike laminar flows where density ratios are unimportant (Benjamin, 1968). In this work, it is
explicitly assumed that the powder avalanche density ratio is sufficiently great that the Boussinesq approximation is not valid. Accordingly, derivation of the model equations must retain the buoyancy fluctuation terms.

Density and Velocity Distributions

Shortly after avalanche release, the velocity and density structure of the flowing avalanche can be described to good approximation by the centre-of-mass theory of Perla et al. (1980). During initial acceleration, the sliding snow mass moves largely with a uniform speed and there is little variation in vertical density (i.e., snow volume fraction) with height. As snow fragmentation and fluidization ensue, vertical variations in the slope-parallel velocity and in the density profile normal to the bed develop.

Once the avalanche has reached the quasi-equilibrium mid-track region, a geometrical structure similar to that shown in Figure 4 is postulated. The avalanche is composed of a highly turbulent, transient head region and a more geometrically stable steady gravity current (or body) region. Here, a finite length to the avalanche, of order $10^{-10}$ meters, is assumed as opposed to the endless fluid approach of Voellmy and other avalanche hydrodynamicists. The slope parallel avalanche velocity is seldom constant; most often the front moves with a wave-like fluctuation, even on slopes of nearly uniform inclination (Van Wijk, 1967).

Within both the head and body regions, the density and velocity distributions attain profiles that resemble other density stratified geophysical flows. The mean avalanche density (or, alternatively, the snow volume fraction) is greatest near the bed; above a dense core which closely follows the track surface, the density profile drops off quickly, attaining a mean value representative of the buoyant powder snow cloud. (See, for example, the experimental studies by Shoda, 1966, and Kotlyakov et al., 1977.) There is some indication that at the front of the avalanche, the density maxima may actually occur slightly off the bed, at the height of the nose (Hopfinger and Tochon-Danguy, 1977). The velocity profile for the slope parallel component increases from zero at the bed surface to a maximum value perhaps a few tenths of the flow depth from the bed. From the maxima, the velocity profile decreases to a near zero value at the interface between the powder cloud and adjacent ambient fluid. The extent to which the avalanche induces a drag on the ambient atmosphere causing a downslope motion in the clear fluid above the avalanche may be explored through numerical simulation experiments.

Plausible vertical distributions of avalanche density and velocity may be inferred from other density stratified geophysical flows including laboratory density and gravity currents (Figures 5 and 6) and density-stratified, partially-mixed estuaries (Tesche 1974, 1975a, b) (Figure 7).

For gravity currents on inclines and presumably avalanches as well, the front velocity is less than that of the following body of the flow. Simpson (1982) gives a typical value of 60% as the fractional front speed. The head volume increases by direct entrainment of snow and air from in front and on the sides, as well as from mass supplied by the faster moving trailing body.

Only limited experimental data are available which examine the cross-slope (y-direction) velocity and density distributions of either avalanches or density currents. Laboratory experiments reported by Luthi (1981) indicate that flow velocities and densities are greatest along the central flow axis and lowest at the outer margins of the current. Fietz and Wood (1967), discussing non-suspension-type gravity

Figure 4. Powder avalanche in the quasi-equilibrium regime indicating snow and air entrainment mechanisms. (Modified from Hopfinger, 1983.)

Figure 5a. Velocity profiles in a gravity current on a slope of 10 degrees, with $\Delta \rho / \rho = 1\%$ ($U_t = 7$ cm/s).

Figure 5b. Density profiles in a gravity current on a slope of 10 degrees with $\Delta \rho / \rho = 1\%$. 
currents, suggest that the cross-slope velocity and density distributions in turbulent flows may be self-preserving (i.e., the distributions vary only in scale from one cross-section to another). This condition may obtain for idealized avalanches on smooth, unconfined tracks, but for irregular, confined flows the cross-stream momentum balance is expected to vary significantly from one section to another due to entrainment, sedimentation, topographic channeling, etc.

Because the avalanche flow is turbulent and transient, the location of maximum frontal velocity will vary from one location to another across the lateral flow coordinate. Gravity current studies by Simpson (1972) have shown the existence of lobe structures across the front of the current. Motion pictures of powder avalanches also reveal stochastic variation in the cross-stream location of maximum frontal velocity (see, for example, the film Avalanche Control distributed by the U.S. Forest Service.).

Air and Snow Entrainment

The powder snow avalanche and its precursor, the dense-snow flowing avalanche entrain both air and snow in complex ways. The exact mechanisms and the relative importance of one versus another is probably specific to a given avalanche. In spite of the obvious complexities, the entrainment processes influencing avalanche motion are amenable to approximation and representation in a numerical model. For the present, postulated air and snow entrainment mechanisms are discussed quantitatively, reserving for later the development of quantitative algorithms for the entrainment-closures of the model equations.

Ambient air entrainment affects the degree of fluidization of the powder avalanche during its initial development (Shreve, 1968). The region of greatest air entrainment may be slightly upslope from the head (Beghin and Brugnot, 1983) in a region where the wall pressure is lowered due to suction (Hopfinger, 1983). While air entrainment tends to reduce the density ratio, $\Delta \rho / \rho$, of the avalanche and the interphase frictional forces (thereby facilitating turbulent suspension), it also has a retarding effect. Entrained air represents a momentum loss to the powder avalanche, the significance of which increases as the mass of entrained air increases and the avalanche density ratio decreases.

Laboratory experiments reveal the strong dependence of the shape of the turbidity current head upon slope angle and entrainment rate, with the head increasing with increasing slope angle. The head entrains ambient fluid by turbulent mixing and it is also supplied from the rear by mass from the quasi-steady body. At larger slope angles there is greater mixing both into the head and into the current behind (Britter and Linden, 1980), leading to an increase in the dimensions of the head with downslope distance.

The rate of growth of the avalanche depth ($dH/dx$) is no doubt a function of turbulence intensity and the degree of suspension of snow particles. The former is believed to be directly related to slope angle. Laboratory experiments by Beghin et al (1981) have shown that the height growth rate of inclined turbulent thermals is linearly related to slope angle, $\theta$. This finding is also confirmed in the experiments of Ellison.
Analysis of debris flow experiments indicates that most of the fluid entrainment takes place along the sloping face of the nose. Just where along the free upper surface air entrainment is greatest in the avalanche is uncertain but most workers appear to favor greatest entrainment of ambient fluid upslope of the head. Most relationships for air entrainment assume that the rate is proportional to mean flow speed. However, Scheiwiller (1986) also includes a pressure and vertical velocity dependence in formulating his entrainment closure.

When the intergranular cohesive strength of the upper snow layer is less than the stress applied to it by the moving avalanche, new snow is dislodged and may be entrained. Snow may be entrained into an avalanche at several different locations including:

- Upward from the bed surface into the avalanche head and/or body,
- Along the forward sloping face of the avalanche front,
- Behind (uphill) the avalanche head,
- At the nose of the avalanche as a consequence of ploughing (Perla, 1980),
- On the lateral flanks of the avalanche.

The relative importance of each of these entrainment possibilities depends upon the development stage of the avalanche and the particular path. Figure 8 illustrates several different air/snow entrainment scenarios for a developing powder avalanche based on the laboratory analyses of Hopfinger and Tochon-Danguy (1977).

Snow entrainment provides both a buoyancy producing and retarding mechanism. Entrainment of new snow increases the flux of negatively buoyant material which contributes to avalanche acceleration. At the same time, however, the influx of low momentum new snow tends to retard avalanche motion until momentum is redistributed throughout the flow.

In models which explicitly account for air entrainment effects (e.g., Hopfinger, 1983; Hopfinger and Tochon-Danguy, 1977; Beghin and Brugnot, 1983), the air entrainment rate, \( E = \frac{\partial h}{\partial x} \), is based upon the experimental results of Ellison and Turner (1959) who give:

\[
E = 10^{-3} (\theta + 5)
\]  

(43)
for situations when the density difference is small. Although the density difference of the avalanche as a whole is not small, in the intermittent region at the top surface of the avalanche, the bulk density of the snow-laden air closely approximates that of air and so this expression is assumed to hold. For high-density flows, the rate of growth is somewhat reduced (Townsend, 1966).

Hopfinger and Beghin (1980) show from theoretical agreements that for large density-difference effects, the right hand side of Equation (43) must be multiplied by \((p/p_d)^{1/2}\). They also suggest that since air entrainment is largely uninfluenced by inertial effects, a high density ratio powder avalanche will lose relatively less momentum to the entrained air (compared to a low density cloud) and will therefore be less affected by air entrainment.

Some quantitative guidance is available with which to formulate the conceptual model of snow entrainment along bottom, front, top, and lateral surfaces of the powder avalanche. Where new snow entrainment is treated in current models, it is assumed to occur at a constant rate along the track (Perla, 1980), or proportional to flow depth (Danilov and Eglit, 1977), or proportional to flow velocity (Eglit, 1983).

Mixing of air into the body of the avalanche is most likely a result of: (1) instabilities occurring at the interface between the avalanche and the air, and (2) fluid turbulence within the flow itself. Which of the two mechanisms is most important is likely to be determined by local avalanche properties. Whether shear layer instability or fluid turbulence is the more effective density current entrainment mechanism has been intensively investigated for several decades at least (see, for example, Middletown, 1966; Morgenstern, 1967; Daly and Pracht, 1968; Allen, 1971; Middletown and Hampton, 1976; Komar, 1977; Griffiths and Hopfinger, 1983; and Wang, 1985). At low turbulence intensities, entrainment of air/snow into the avalanche may be due principally to interfacial shear instability near the front of the gravity current (Turner, 1972; Britter and Simpson, 1978) whereas for flow with large turbulence intensities (>50%), the entrainment is most probably maintained by eddy impingement on the interface (Thomas and Simpson, 1985).

Sedimentation and Avalanche Termination

Removal of suspended snow particles from the avalanche through sedimentation is closely associated with avalanche deceleration and termination. When the shear production of turbulent kinetic energy is reduced to a point where the turbulence intensity is insufficient for maintenance of suspended snow particles, sedimentation will occur. Location of those regions within the moving avalanche where sedimentation is most efficient depends upon the turbulence levels. Following Britter and Simpson (1978) and Simpson and Britter (1979), the turbulence intensity in the body of the avalanche is assumed to be less than in the head and as a consequence, snow particle sedimentation begins initially in the body region. As avalanche buoyancy is reduced through sedimentation, turbulence levels diminish, feeding back to an increased sedimentation rate. Hopfinger (1983) characterizes this situation as one in which the head of the avalanche becomes cut off from its buoyancy supply; the head then behaves as a turbulent cloud which disperses downslope. In sum, the particle sedimentation process is governed principally by the size distribution of the suspended phase and the local fluid turbulence level (Graf, 1971).

Based upon the foregoing conceptualization of the powder avalanche phenomena, the mathematical formulation of the model can now be developed.

4.0 DERIVATION OF THE CONSERVATION EQUATIONS FOR MULTIPHASE TURBULENT AVALANCHE FLOW

The mathematical description of avalanche dynamics is based upon the laws of mass, momentum, and energy conservation. The conservation equations derived in this section are for a Newtonian, non-Boussinesq, multiphase turbulent fluid.

Navier-Stokes Equations

The conservation of mass law applied to a fluid passing through an infinitesimal, fixed control volume yields the continuity equation:

\[
\frac{\partial \rho u}{\partial t} + \frac{\partial }{\partial x} (\rho u^2 + p - \tau_{xy}) + \frac{\partial }{\partial y} (\rho u v - \tau_{yx}) + \frac{\partial }{\partial z} (\rho u w - \tau_{xz}) = \rho g_x
\]

where \(p\) is the fluid density and \(u, v,\) and \(w\) represent the \(x, y,\) and \(z\) components of the velocity vector. A right hand coordinate system as shown in Figure 4 is adopted.

The momentum equation is obtained by applying Newton’s Second Law to the fluid:

\[
\frac{\partial \rho u}{\partial t} + \frac{\partial }{\partial x} (\rho u^2 + p - \tau_{xy}) + \frac{\partial }{\partial y} (\rho u v - \tau_{yx}) + \frac{\partial }{\partial z} (\rho u w - \tau_{xz}) = \rho g_x
\]

\[
\frac{\partial \rho v}{\partial t} + \frac{\partial }{\partial x} (\rho u v - \tau_{xy}) + \frac{\partial }{\partial y} (\rho v^2 + p - \tau_{yy}) + \frac{\partial }{\partial z} (\rho v w - \tau_{yz}) = \rho g_y
\]

\[
\frac{\partial \rho w}{\partial t} + \frac{\partial }{\partial x} (\rho u w - \tau_{xz}) + \frac{\partial }{\partial y} (\rho w v - \tau_{yx}) + \frac{\partial }{\partial z} (\rho w^2 + p - \tau_{zz}) = \rho g_z
\]

where the components of the viscous stress tensor \(\tau_{ij}\) are given by:

\[
\tau_{ij} = -\rho \frac{\partial u_i}{\partial x_j} - \rho \frac{\partial u_j}{\partial x_i} + \frac{\partial}{\partial x_j} \left( \frac{2}{3} \frac{\partial u_i}{\partial x_k} \frac{\partial u_k}{\partial x_j} \right)
\]
Because the last two terms in Equation (53) when combined, equal zero, the continuity equation in mass-weighted variables may be written as:

\[ \rho \overline{f''} = 0 \]  

(52)

With the above definitions, the mass-weighted Reynolds equations can now be developed. Substituting the mass-weighted averaged variables plus the doubly primed fluctuations given by Equation (51) into Equation (44) and time averaging the entire equation gives:

\[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} (\rho \bar{u}_j) = \frac{\partial}{\partial x_j} (\rho \bar{u}_j u''_j) + \frac{\partial}{\partial x_j} (\rho' \bar{u}_j) = 0 \]  

(53)

Because the last two terms in Equation (53) when combined, equal zero, the continuity equation in mass-weighted variables may be written as:

\[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} (\rho \bar{u}_j) = 0 \]  

(54)

The Reynolds form of the momentum equation can be developed by direct substitution of the velocity decompositions [Equation (51)] into Equations (45-48). Time averaging the entire vector equation gives:

\[ \frac{\partial}{\partial t} (\rho \bar{u}_i) + \frac{\partial}{\partial x_j} (\rho \bar{u}_i \bar{u}_j) - \frac{\partial \rho}{\partial x_i} \]  

(55)

where, neglecting viscosity fluctuations, \( \tau_{ij} \) becomes:

\[ \tau_{ij} = \mu \left[ \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right] - 2 \delta_{ij} \frac{\partial \bar{u}_k}{\partial x_k} \]  

(56)

In general, the viscous terms involving the doubly primed fluctuations are expected to be small and may often be dropped based upon order of magnitude analysis.

The Reynolds equations (Equations 54 and 55) apply to the dynamics of a turbulent, nonsteady, Newtonian single phase fluid. They cannot be solved in the form given, however, because of the introduction of new apparent turbulent stress quantities. To proceed further, it is necessary to formulate additional constitutive equations for the turbulent quantities (based upon conservation laws) or postulated relationships between the new apparent turbulent quantities and the
time-mean flow variable. This is the classical closure problem and it is typically attacked through some form of turbulence modeling. Before developing the turbulence closure equations for the avalanche dynamics model, though, the Reynolds equations are first extended to multi-phase avalanche flow. Specifically, conservation equations for air and for suspended snow particle phases of the avalanche are written individually. Then a combined relation governing the bulk flow (e.g., for the turbulent suspension of snow in air) is developed.

Multi-Phase Flow Equations for Powder Avalanche

Turbulent avalanche flow is a two-phase transport phenomena. Rigorous mathematical description requires that the conservation relations for mass and momentum be written for both the motion of the carrier fluid (air) as well as the motion of the suspended snow particle phase. While it is conceptually possible to follow the motion of each discrete snow particle (Perl et al., 1984; Greenspan, 1985) computational constraints generally dictate that some form of volume averaging and/or statistical representation of the suspension be employed. In much the same manner that computational restrictions necessitate time-averaging to make the Navier-Stokes equations tractable, for multiphase flow volume averaging is needed to address computational constraints.

The avalanche is conceptualized as a turbulent suspension of snow particles as shown diagrammatically in Figure 9. The gas phase, $\alpha$, is the carrier fluid within which solid snow crystals (the $\beta$ phase) are suspended through turbulent motion. The total volume, $\Gamma$, consists of the fluid component, $\Gamma_f$, and the solid component, $\Gamma_s$. The snow particle volume fraction is defined as:

$$f_\beta = \frac{\Gamma_f}{\Gamma} \quad (57)$$

Following Soo (1967), the conservation of mass equation for the fluid is:

$$\frac{\partial \rho_\alpha}{\partial t} + \frac{\partial}{\partial x_i} \left( \rho_\alpha \bar{u}_i \right) = 0 \quad (58)$$

This equation may be volume averaged (Gray, 1975; Gray and Lee, 1977; Slattery, 1981; Whittaker, 1969) to give:

$$\frac{\partial}{\partial t} \left\langle \rho_\alpha \right\rangle + \frac{\partial}{\partial x_i} \left\langle \rho_\alpha \bar{u}_i \right\rangle = 0 \quad (59)$$

Hereafter, the angle brackets $\langle \rangle$ are used to represent a volume average.

The volume averaged conservation of momentum equation for the air phase including the gravity term is written as:

$$- \frac{\partial}{\partial t} \left\langle \rho_\alpha \bar{u}_i \bar{u}_j \right\rangle + \frac{\partial}{\partial x_i} \left( \rho_\alpha \bar{u}_i \bar{u}_j \right) = - \frac{\partial}{\partial x_i} \left\langle \rho_\alpha \bar{u}_i \phi_{ij} \right\rangle \quad (60)$$

Here, the superscripts $\alpha$ and $\beta$ refer to the phase over which the volume averaging is performed. The single primed quantities are deviations from the volume average and will be treated in a manner analogous to the Reynolds stresses. The volume averaged momentum equation for the $\alpha$-phase thus becomes:

$$- \frac{\partial}{\partial t} \left[ \left( \rho_\alpha \bar{u}_i \bar{u}_j \right) \right] + \frac{\partial}{\partial x_i} \left[ \rho_\alpha \bar{u}_i \bar{u}_j \right] = - \frac{\partial}{\partial x_i} \left[ \rho_\alpha \bar{u}_i \phi_{ij} \right] \quad (62)$$

Here, the superscripts $\alpha$ and $\beta$ refer to the phase over which the volume averaging is performed. The single primed quantities are deviations from the volume average and will be treated in a manner analogous to the Reynolds stresses. The volume averaged momentum equation for the $\alpha$-phase thus becomes:

$$\alpha$$-Momentum Equation

$$- \frac{\partial}{\partial t} \left[ \rho_\alpha \left( \bar{u}_i \bar{u}_j \right) \right] + \frac{\partial}{\partial x_i} \left[ \rho_\alpha \bar{u}_i \bar{u}_j \right] = - \frac{\partial}{\partial x_i} \left[ \rho_\alpha \bar{u}_i \phi_{ij} \right] \quad (62)$$

Here, the superscripts $\alpha$ and $\beta$ refer to the phase over which the volume averaging is performed. The single primed quantities are deviations from the volume average and will be treated in a manner analogous to the Reynolds stresses. The volume averaged momentum equation for the $\alpha$-phase thus becomes:

$$\alpha$$-Momentum Equation

$$- \frac{\partial}{\partial t} \left[ \rho_\alpha \left( \bar{u}_i \bar{u}_j \right) \right] + \frac{\partial}{\partial x_i} \left[ \rho_\alpha \bar{u}_i \bar{u}_j \right] = - \frac{\partial}{\partial x_i} \left[ \rho_\alpha \bar{u}_i \phi_{ij} \right] \quad (62)$$

Here, the superscripts $\alpha$ and $\beta$ refer to the phase over which the volume averaging is performed. The single primed quantities are deviations from the volume average and will be treated in a manner analogous to the Reynolds stresses. The volume averaged momentum equation for the $\alpha$-phase thus becomes:
where $S_{abij}$ and $\phi_{abij}$ are surface stress terms arising from the interaction between air and snow particle motion.

The equation for momentum conservation for the $\beta$-phase (snow) is developed by analyzing the ensemble motion of single snow particles. For a nondeformable, nonrotational snow particle, the time rate of change of its momentum is equal to the sum of the surface shear and body forces,

$$\frac{d}{dt}(m \ddot{u}_{\beta i}) = \tau_{\beta i} + m \dot{g}_{\beta i} \quad \text{(no sum on p)} \quad (63)$$

where $m$, $\ddot{u}_{\beta i}$, and $\tau_{\beta i}$ are the particle mass, velocity, and surface shear stress, respectively. Because particle mass and density are fixed until the latter stage of the avalanche where sedimentation and densification occur, the body force may be expressed as:

$$\langle m \dot{g}_{\beta i} \rangle = f_{\beta} p \rho_{\beta} \dot{g}_{\beta i} \quad (64)$$

Furthermore, the surface shear stress term, $\tau_{\beta i}$, must be equal and opposite to a corresponding surface force acting on the fluid, $\tau_{\beta i}$. This is convenient, for in the subsequent derivation of the bulk flow equations, the surface terms cancel.

The $\beta$-phase velocity is defined as the volume-averaged momentum in the $\beta$-phase divided by its mass so that:

$$\bar{u}_{\beta i} = \frac{\langle m \ddot{u}_{\beta i} \rangle}{\langle m \ddot{u}_{\beta i} \rangle} \quad (65)$$

Assuming sufficient snow particles to form a continuum, the volume-averaged momentum equation for the snow phase may be written as:

**$\beta$-Momentum Equation**

$$-\frac{\partial}{\partial t} \langle f_{\beta} \ddot{u}_{\beta i} \rangle + \frac{\partial}{\partial x_j} \langle f_{\beta} \rho_{\beta} \ddot{u}_{\beta j} \ddot{u}_{\beta j} \rangle = -S_{abij} + f_{\beta} \rho_{\beta} \dot{g}_{\beta i} - \frac{\partial}{\partial x_j} \phi_{abij} \quad (66)$$

Similarly, because a continuum is assumed for the $\beta$-phase, the volume averaged continuity equation for snow may be written as:

**Snow Continuity Equation**

$$\frac{\partial}{\partial t} \langle p_{\beta} \rangle + \frac{\partial}{\partial x_j} \langle f_{\beta} \rho_{\beta} \ddot{u}_{\beta j} \rangle = 0 \quad (67)$$

Three terms in the momentum equations (Equations 62 and 66) arise as a consequence of volume averaging. As mentioned above, terms involving $S_{abij}$ and $\phi_{abij}$ represent surface momentum exchanges occurring at the local boundaries separating the air and snow particle phases. The last term in Equation 62 is analogous to the Reynolds stress terms which arose earlier due to temporal averaging; here, these terms arise as a result of volume averaging.

Equations 62, 66, and 67 describe the motion of multi-phase avalanche flow at any instant for a particular averaging volume which is large with respect to particle size yet small with respect to the characteristic length scales of the avalanche. Principal interest is in knowing the bulk flow properties of the avalanche (i.e., mean velocity, pressure, mass, and energy distributions). Since the surface momentum exchange terms, $S_{abij}$ and $\Gamma_{abij}$, are poorly understood given current observational data for avalanches, it is convenient to sum Equations 62 and 66, thereby eliminating the interfacial exchange terms and producing conservation relations for the bulk flow properties. Note that the model developed by Scheiwiller and co-workers does not involve adding the two phase relations, but instead entails the specification of constitutive relationships for inter-phase effects. In the present model development, some generality is sacrificed in developing bulk flow equations in order to extend the modeling system to three dimensions.

**Summation and rearranging terms in Equations 62 and 66 yields:**

**Bulk Continuity Equation**

$$\frac{\partial}{\partial t} \langle p_{\beta} \rangle + \frac{\partial}{\partial x_j} \langle f_{\beta} \rho_{\beta} \ddot{u}_{\beta j} \rangle = 0 \quad (68)$$

and

**Bulk Momentum Equation**

$$\frac{\partial}{\partial t} \langle f_{\beta} \rho_{\beta} \ddot{u}_{\beta i} \rangle + \frac{\partial}{\partial x_j} \langle f_{\beta} \rho_{\beta} \ddot{u}_{\beta i} \ddot{u}_{\beta j} \rangle = \frac{1}{\rho_a} \frac{\partial}{\partial x_j} \langle f_{\beta} \rho_{\beta} \ddot{u}_{\beta i} \ddot{u}_{\beta j} \rangle - \frac{\partial}{\partial x_j} \left[ (1-\bar{\rho}_b) \langle \ddot{u}_{\beta 1} \ddot{u}_{\beta 1} \rangle \right]$$

where $u$ is the snow particle settling velocity and is a function of particle size and local turbulence intensity of the avalanche.

The subscript $b$ refers to the bulk flow properties. Also, the mass continuity equation for snow ($\beta$ phase) becomes:

**Snow Conservation Equation**

$$\frac{\partial}{\partial t} \langle \rho_{\beta} \rangle + \frac{\partial}{\partial x_j} \langle f_{\beta} \rho_{\beta} \ddot{u}_{\beta j} \rangle = \frac{\partial}{\partial x_j} \langle f_{\beta} (1-\bar{\rho}_b) \ddot{u}_{\beta j} \rangle + \frac{\partial}{\partial x_j} \langle \rho_b \ddot{u}_{\beta 1} \ddot{u}_{\beta 1} \rangle \quad (69)$$

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For the present, it is assumed that the turbulent Schmidt number is unity. That is:
\[ K_m = K_s \]  
where \( K_s \) is the eddy mass diffusivity for snow defined analogously to Equation (73).

Generalized turbulent transport equations for turbulent kinetic energy and dissipation are written as follows:

**Kinetic Energy Transport Equation**
\[
\frac{\partial k}{\partial t} + \sum_{i} \frac{\partial k}{\partial x_i} = \frac{\partial}{\partial x_i} \left[ \frac{K_m}{\sigma_k} \frac{\partial k}{\partial x_i} \right] + Pr - \varepsilon + G
\]  

and

**Dissipation Transport Equation**
\[
\frac{\partial \varepsilon}{\partial t} + \sum_{i} \frac{\partial \varepsilon}{\partial x_i} = \frac{\partial}{\partial x_i} \left[ \frac{K_m}{\sigma_\varepsilon} \frac{\partial \varepsilon}{\partial x_i} \right] + C_{\varepsilon 1} \varepsilon \left[ Pr + (1-C_{\varepsilon 3}) G \right] - C_{\varepsilon 2} \frac{\varepsilon^2}{k}
\]  

where \( Pr = \frac{u'' w''}{\rho_0} \frac{\partial \bar{u}}{\partial x_i} \) is the shear-induced production of the Reynolds stress, and
\[ G = (\bar{\rho} - \rho_0) u_j \lambda \]  
is the buoyancy-induced production term. The turbulent kinetic energy and dissipation are defined as:
\[ k = \frac{1}{2} u''_{i j} \]  
and
\[ \varepsilon = \nu \left[ \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \bar{u}_j}{\partial x_i} \right] \]  

where
\[ u''_{i j} = K_m \left[ \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right] - \frac{2}{3} k \delta_{i j} \]  

C depends on the density stratification and surface damping:
\[ C_\mu = \frac{2}{\mu k} \]
\[
\omega = \left(1 - \frac{C_2 + \frac{3}{2} C_2 C_2'}{C_1 + \frac{3}{2} C_1'}\right)^{-1} \left(1 - \frac{1}{C_1 + \frac{3}{2} C_1'}\right) \left(1 - \frac{C_2 T}{C_1 T}\right) \alpha B
\]

\[
a = \frac{1}{C_1 T + C_1'} f + 2(1 - C_3 T) R B
\]

\[
B = \beta g \frac{k^2}{\varepsilon} \frac{\partial T}{\partial y}
\]

\[
\frac{\omega_k^2}{\kappa_1} = \frac{2}{3} \left[ \Omega_1 + \Omega_2 \right]
\]

where:
\[
\Omega_1 = \left[ C_1 - 1 + \frac{P+G}{\varepsilon} \left( C_2 - 2C_2 C_2' f \right) \right]
\]

\[
\Omega_2 = \left[ \frac{G}{\varepsilon} \left( C_1 - 2C_2 + 2C_2 C_2' f \right) \right] \left( C_1 + 2C_1' f + \frac{P+G}{\varepsilon} - 1 \right)
\]

Finally,
\[
\sigma_k = \frac{\omega}{C_k}
\]

\[
\sigma_\varepsilon = \frac{\omega}{C_\varepsilon}
\]

For the constants appearing in the above buoyancy extensions of the \(k-\varepsilon\) model, the following values have been adopted from Gibson and Launder (1978):

<table>
<thead>
<tr>
<th>(C_k)</th>
<th>(C_\varepsilon)</th>
<th>(C_1)</th>
<th>(C_2)</th>
<th>(C_3)</th>
<th>(C_1')</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.24</td>
<td>0.15</td>
<td>2.2</td>
<td>0.55</td>
<td>0.55</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Equations 68, 69, 70, 76, and 77 collectively represent the conservation equations for the three-dimensional powder avalanche model. Written in primitive variable form, these equations and their associated constitutive relationships and definitions can be integrated numerically to give the time-dependent, spatially-varying velocity and mass density fields. The following two sections discuss the boundary and initial conditions necessary to solve the governing equations by numerical methods.

**BOUNDARY CONDITIONS**

Since the governing relations for the powder avalanche are differential equations, complete problem specification must include boundary conditions. Several boundary conditions must be specified for solution of Equations 68-72. For this model, specification of conditions at the erodable bed surface, the lateral (y-direction) boundary and the free surface aloft are most important. Determination of boundary conditions at boundaries where fluid enters or exits or along a free surface boundary is straightforward for most of the variables. In these cases, either the variable is known or a known (often zero) or estimated flux can be prescribed.

Free surface fluid boundaries are treated as planes of symmetry for velocities and the turbulence variable \(k\) and \(\varepsilon\). Along the free surface there is no assumed snow mass flux, so:

\[
\bar{f}_\beta \left(1 - f_b\right) \bar{u}_s + \bar{f}_\beta K_m \frac{\partial \bar{f}_\beta}{\partial x} + \frac{\partial \bar{f}_\beta}{\partial y} = 0 \quad z = z_s
\]

where \(K_m\) is the turbulent eddy viscosity. A similar relationship is assumed along the bed surface, where a fraction \(D\) of snow reaching the wall is resuspended by local turbulence; thus:

\[
\bar{f}_\beta \left(1 - f_b\right) \bar{u}_s + D \bar{f}_\beta K \left[ \frac{\partial \bar{f}_\beta}{\partial x} + \frac{\partial \bar{f}_\beta}{\partial y} \right] + E = 0 \quad z = 0
\]
where \( E_0 = E(x,y,t) \) is the bed surface snow entrainment rate.

Since the powder avalanche is a wall bounded turbulent flow (bounded by the ground-snow surface), the turbulent closure scheme presented in the previous section does not apply in the near vicinity of the bed surface. Accordingly, a matching scheme is needed to treat the viscous sublayer. This is accomplished using the law of the wall boundary condition which is valid out to a distance \( \delta \) from the bed. The boundary condition for a hydraulically rough surface is:

\[
    u = \frac{u_*}{k} \ln \left( \frac{k \delta}{e} \right) \quad (92)
\]

where \( k \) is the von Karman constant, \( e \) is the characteristic roughness height, and \( u_* \) is the friction velocity. Recent analyses by Coleman (1981, 1986) indicate that for sediment-laden flows, the value of \( k=0.44 \) is essentially constant over a range of Richardson numbers from zero (clear fluid) to about 10\(^{-2} \) (capacity sediment suspension). Coleman also reports that for suspended-sediment flow, the logarithmic portion of the velocity profile is shifted downward (toward the bed) relative to homogeneous flows.

Wall boundary conditions for turbulent kinetic energy and dissipation are:

\[
    k(a) = \frac{u_*^2}{(C_f)1/2} \quad (93)
\]

\[
    \varepsilon(a) = \frac{u_*^3}{k a} \quad (94)
\]

The latter expressions derive from the assumption that near the surface of the avalanche (in the constant stress layer) kinetic energy production is balanced by dissipation.

6.0 NUMERICAL SOLUTION PROCEDURES

The time-dependent set to be solved for the powder avalanche is a mix of elliptic-parabolic coupled partial differential equations. Subject to certain simplifications, the Reynolds averaged form of the Navier-Stokes equations must be integrated with a powerful, computationally efficient numerical technique. This section discusses considerations related to the selection of appropriate numerical methods.

A major consideration is which generic type of numerical approach to employ—particle, finite difference, finite element, or hybrid. The recent literature in computational fluid dynamics contains a number of excellent articles presenting the advantages and drawbacks of the alternative methods for integrating the turbulent Navier-Stokes equations (see for example, Greenspan, 1985; Van de Vosse et al., 1986; Connell and Stow, 1986; Theodossiou and Sousa, 1986; and Fulton et al., 1986). While finite difference and finite element methods have reached a high level of sophistication and diversity, other methods (e.g., particle or stochastic) may be more appropriate choices in situations where the continuum assumption becomes tenuous, where numerical diffusion errors become significant, or where sub-grid scale turbulence phenomena cannot be adequately parameterized. In practice, selection of a numerical method has become often a matter of taste and expediency. For the present, the multi-grid finite difference method is selected but hybrid techniques may prove to be more attractive once further experience with numerical integration of the field equations is gained.

Linearization

Solution of the model equations is usually obtained by transforming them into analogous algebraic equations. Since both the governing equations and their simpler algebraic analogs are nonlinear, it is essential to linearize them in order to obtain economical solutions. The nature of the linearization largely determines the form of the numerical technique that is subsequently used. For example, both coupled and uncoupled iterative schemes can be developed for the Navier-Stokes equations depending upon the linearization approach selected.

Obviously, the type of linearization can significantly affect the rate of convergence of the numerical method. Several alternatives for linearization presently exist. One is the popular segregated solution method (Patankar, 1980). Another involves a full Newton-Raphson linearization of all non-linear terms (Galpin and Raithby, 1986). While the latter method has been shown to enhance convergence, especially in flow configurations with grid curvature (important for avalanche simulation), the method is computationally burdensome. If a direct solution procedure such as outlined by Galpin and Raithby is used, it appears appropriate to consider an iterative scheme using a coupled-equation line solver to implicitly retain the inter-equation couplings.

Galpin and Raithby (1986) suggest that these alternative methods are competitive in terms of computational efficiency. Along similar lines, the modified pressure, implicit predictor-corrector scheme combined with a multi-grid solution method developed by Theodossiou and Sousa (1986) shows promise.

Differencing Methods

For the three dimensional problem under consideration, the standard vorticity-stream function approach to solving the Navier-Stokes equations becomes intractable because a single stream function does not exist (Anderson, et al., 1984; Yih, 1957). Therefore, the field equations must be solved in their primitive variable form \((u,v,w, \text{ and } p)\).

Numerical solution of the multi-phase, multi-dimensional equation set is expected to tax
the computing power of all but the largest vector CPU's available today. As a result, judicious selection of the discretization scheme to be employed is necessary. Recent work by Patel and Markatos (1986) provides insight. They evaluated eight different discretization schemes for their generality, relative accuracy, stability, and computational requirements. From their study results, it appears that one of the contemporary quadratic upstream extended differencing schemes offers the best compromise between accuracy and cost. Since their simulations were confined to two dimensional, low Reynolds number flows, it is likely that experimentation with alternative schemes will be necessary in extending the techniques to three dimensions and turbulent flow.

Multi-Grid Solution Method

Multi-grid methods have been developed over the last decade to solve a large class of problems in geophysical fluid dynamics. They are particularly attractive in the solution of boundary value elliptic problems, especially in the case of three dimensional flows where an efficient, time-dependent solution procedure is imperative.

The basic concept behind the multi-grid approach is to approximate the same governing equations on a set of overlapping uniform grids of widely varying mesh sizes and to iteratively cycle between these discrete problems to produce the solution on the finest grid possible (Brandt, 1984; Fulton et al., 1986). Multi-grid methods thus differ from nested grid techniques in that the latter seek to provide finer resolution in flow regions close to solid boundaries, interfaces, etc. The primary advantages of multi-grid methods are efficiency and generality. However, the operational complexity of the multi-grid method makes this approach significantly more difficult to implement compared with other methods.

Currently, there is a growing supply of algorithms and computer codes available on multi-grid methods for geophysical fluid dynamic problems (Brandt, 1982, 1984; Foerster and Witsch, 1982; Dendy, 1982,1983; and Rice, 1984). Despite its complexity, the multi-grid method appears to be the most attractive technique for iteratively solving the powder avalanche finite difference equations. The approach is powerful, flexible, and some general computer software packages written specifically for vector and parallel processor computers such as the CRAY-1, CRAY-XMP, and CYBER 205 are becoming available. Finally, recent developments in sophisticated discretization techniques (i.e. Multi-Level Adaptive Techniques) for elliptic geophysical flow problems show great promise when combined with multi-grid methods (Fulton et al., 1986).

Grid Mesh Generation

Numerical solution of the governing equations is greatly simplified by a well-constructed grid. Poor grid specification can lead to numerical instabilities or slow convergence. One powerful method of grid construction is to map the complex physical flow field of the avalanche into a computational domain of far greater simplicity (Figure 10). Advantages of such a mathematical transformation includes (1) the avalanche track surface (below erodable snow cover) can be selected as the boundary in the computational plane permitting easy application of surface boundary conditions, (2) unequally spaced points in the physical plane can be mapped into a uniformly spaced grid in the computational domain, and (3) if a differential equation is used to generate the grid, the properties of the solution of the grid generating equation can be used to directly aid in specifying the mesh.

Of the various types of partial differential equation grid generation schemes available (see for example Anderson et al., 1984), the recent scheme proposed by Steger and Sorenson (1980) seems to be well suited to avalanche flow simulation. Their method uses a system of hyperbolic equations to generate a grid mesh over a solid surface where the outer flow boundary need not be specified a priori. This is convenient in the free surface avalanche flow problem since the upper boundary location is transient. In Steger and Sorenson's method, the bed surface forms the inner boundary and the hyperbolic system of equations is marched out in the hyperbolic direction. A volume or a gravity scheme is used to prescribe the mesh.

Figure 10. Conceptual scheme for mapping the physical domain of the powder avalanche to the simplified computational domain. (Only one realization of the computational grid mesh is depicted.)

Initialization

The three phases of the powder snow avalanche—acceleration, quasi-equilibrium, and runout—are modeled as discrete sequences. The acceleration phase, consisting of initial release followed by accelerating flow, can be treated as a two-dimensional, time-dependent laminar flow. The governing field equations for this initial phase of the flow are:
These equations are integrated from release to the point where transition from a laminar, flowing avalanche to a turbulent powder avalanche occurs. At this point, the velocity, mass, and kinetic energy fields are mapped into an initial field which is used to initialize the model equations for the powder avalanche (Equations 68-70, 76 and 77).

Solution to the above equation set for the laminar-transition flow problem is readily obtained through modifications to the AVALANCHE code. Initial sensitivity testing of the refined code indicates that reasonable mean flow heights and speeds can be estimated provided appropriate choices of model parameters are made (Tesche, 1986). For example, in test simulations of a large avalanche (vertical drop of 350 meters along a 1,000 meter track) variation in the surface friction coefficient between 0.25 and 0.75 had little effect on the time to maximum flow velocity, but the velocity maxima was reduced by a factor of two (i.e., from 42 m/s to 23 m/s. See Figure 11.

Initial testing of the powder avalanche model with a two-dimensional version of the code will greatly facilitate study of the flow dynamics and the sensitivity of various parameters. Not only are two dimensional models computationally efficient, but their results can generally be quantitatively correct, and are often quantitatively accurate (Heinrich et al., 1983).

7.0 SUMMARY

The phenomena of snow avalanche challenges engineers, geophysicists, and planners who deal with the uncertainties of when, where, and how large avalanches will occur. Because avalanches encompass such a broad range of physical flow conditions, no unified theory of avalanche motion or broadly applicable modeling procedures have yet been advanced. Significant theoretical and
experimental progress has led, however, to a reasonably solid understanding of the taxonomy of snow avalanche. For each general class of avalanche, techniques have been developed for estimating dynamic features of the flow.

A new mathematical formulation to represent turbulent powder avalanches has been developed in the research described in this paper. The approach is based solidly upon physical principles and avoids certain simplifying assumptions, traditionally made, that are actually inappropriate for the density-varying flow phenomena. Most notably, the powder avalanche is considered non-Boussinesq since the density ratios of the avalanche are much larger than unity. The powder avalanche is treated as a two-phase flow phenomenon, with the motion of snow particles and the carrier fluid both considered.

Model development begins with separate mass and momentum conservation laws for the fluid and solid snow fractions and local volume averaging on each set of relations is made. A Reynolds decomposition for turbulent flow is also made. Some simplification is achieved by considering the behavior of a continuum bulk fluid. Addition of momentum and continuity equations for snow and air phases yields these bulk flow relations (Equations 68–70). The model is fully three-dimensional; however, order of magnitude analysis of individual terms in the vector equations will identify terms which may be neglected.

Adaptation of a recent second order closure model (Rodi, 1985) provides a means of estimating turbulent eddy diffusivities for mass and momentum. Model formulation is completed by prescription of boundary conditions for the velocity and mass transport fields. Finally, issues related to the selection of numerical integration methods are introduced. A multigrid finite difference scheme is selected for solving the model equations on a terrain-following grid mesh. Results of initial tests of the initialization scheme based upon a modified version of the AVALANCHE code (developed by Lang and co-workers for laminar, surface avalanches) are presented. In the next series of papers, the new model will be applied to two-dimensional and simplified three-dimensional avalanche path geometries and results of model sensitivity analyses discussed.

ACKNOWLEDGEMENTS

The author gratefully acknowledges the dedication and creativity of Ms. Lynn Nero in skillfully producing this manuscript.

REFERENCES


